CSE 332: Data Abstractions

P, NP, NP-Complete
(part 1)

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Agenda (for next 2 lectures)

• A Few Problems:
  – Euler Circuits
  – Hamiltonian Circuits
• Intractability: P and NP
• NP-Complete
• What now?
Try it!

Which of these can you draw (trace all edges) without lifting your pencil, drawing each line only once? Can you start and end at the same point?
Your First Task

• Your company has to inspect a set of roads between cities by driving over each of them.

• Driving over the roads costs money (fuel), and there are a lot of roads.

• Your boss wants you to figure out how to drive over each road exactly once, returning to your starting point.
Euler Circuits

• **Euler circuit**: a path through a graph that \textit{visits each edge exactly once} and \textit{starts and ends at the same vertex}

• Named after Leonhard Euler (1707-1783), who cracked this problem and founded graph theory in 1736

• An **Euler circuit** exists \textit{iff}
  – the graph is connected and
  – each vertex has \textit{even} degree (\(=\ #\) of edges on the vertex)
The Road Inspector: Finding Euler Circuits

Given a connected, undirected graph $G = (V,E)$, find an Euler circuit in $G$

Can check if one exists:
• Check if all vertices have even degree

Basic Euler Circuit Algorithm:
1. Do an edge walk from a start vertex until you are back to the start vertex.
   – You never get stuck because of the even degree property.
2. “Remove” the walk, leaving several components each with the even degree property.
   – Recursively find Euler circuits for these.
3. Splice all these circuits into an Euler circuit

Running time?
Given a connected, undirected graph $G = (V, E)$, find an Euler circuit in $G$

Can check if one exists: (in $O(|V| + |E|)$)
- Check if all vertices have even degree

Basic Euler Circuit Algorithm:
1. Do an edge walk from a start vertex until you are back to the start vertex.
   - You never get stuck because of the even degree property.
2. “Remove” the walk, leaving several components each with the even degree property.
   - Recursively find Euler circuits for these.
3. Splice all these circuits into an Euler circuit

Running time? $O(|V| + |E|)$
Euler Circuit Example

Euler(A) :
Euler Circuit Example

Euler(A) :
A B G E D G C A
Euler Circuit Example

Euler(A) :  A B G E D G C A

Euler(B)
Euler Circuit Example

Euler(A) : A B G E D G C A

Euler(B) : B D F E C B
Euler Circuit Example

Euler(A) :
A B G E D G C A

Euler(B) :
B D F E C B

Splice

A B D F E C B G E D G C A
Your Second Task

• Your boss is pleased…and assigns you a new task.
• Your company has to send someone by car to a set of cities.
• The primary cost is the exorbitant toll going into each city.
• Your boss wants you to figure out *how to drive to each city exactly once*, *returning in the end to the city of origin.*
Hamiltonian Circuits

- **Euler circuit**: A cycle that goes through each *edge* exactly once.
- **Hamiltonian circuit**: A cycle that goes through each *vertex* exactly once.

Does graph **I** have:
- An Euler circuit?
- A Hamiltonian circuit?

Does graph **II** have:
- An Euler circuit?
- A Hamiltonian circuit?

Which problem sounds harder?
Finding Hamiltonian Circuits

- **Problem**: Find a Hamiltonian circuit in a connected, undirected graph $G$

- **One solution**: Search through *all paths* to find one that visits each vertex exactly once
  - Can use your favorite graph search algorithm to find paths
- This is an *exhaustive search* ("brute force") algorithm

- Worst case: need to search all paths
  - How many paths??
Analysis of Exhaustive Search Algorithm

Worst case: need to search all paths
  – How many paths?
Can depict these paths as a search tree:

Search tree of paths from B
Analysis of Exhaustive Search Algorithm

• Let the *average* branching factor of each node in this tree be $b$

• $|V|$ vertices, each with $\approx b$ branches

• Total number of paths $\approx b \cdot b \cdot b \ldots \cdot b$

• Worst case $\rightarrow$

*Search tree* of paths from B
Analysis of Exhaustive Search Algorithm

- Let the average branching factor of each node in this tree be $b$
- $|V|$ vertices, each with $\approx b$ branches
- Total number of paths $\approx b \cdot b \cdot b \ldots \cdot b = O(b^{|V|})$
- Worst case $\Rightarrow$ Exponential time!

Search tree of paths from B
# More Running Times

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.

<table>
<thead>
<tr>
<th>n</th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
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<td>10</td>
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<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>4 sec</td>
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<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>18 min</td>
<td>$10^{25}$ years</td>
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<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
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<td>36 years</td>
<td>very long</td>
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<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>$10^{17}$ years</td>
<td>very long</td>
</tr>
<tr>
<td>1,000</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>10,000</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>2 min</td>
<td>12 days</td>
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<td>very long</td>
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<tr>
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<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>
Polynomial vs. Exponential Time

• All of the algorithms we have discussed in this class have been **polynomial time** algorithms:
  • Examples: $O(\log N)$, $O(N)$, $O(N \log N)$, $O(N^2)$
  • Algorithms whose running time is $O(N^k)$ for some $k > 0$

• **Exponential time** $b^N$ is asymptotically worse than any polynomial function $N^k$ for any $k$
The Complexity Class P

• $P$ is the set of all problems that can be solved in *polynomial time worst case time*
  – All *problems* that have some *algorithm* whose running time is $O(N^k)$ for some $k$

• Examples of problems in $P$: sorting, shortest path, Euler circuit, *etc.*
P

Sorting
Shortest Path
Euler Circuit
P

Sorting
Shortest Path
Euler Circuit

Hamiltonian Circuit
Satisfiability (SAT)
Vertex Cover
Travelling Salesman
Satisfiability

\((\neg x_1 \lor x_2 \lor x_4) \land (x_1 \lor \neg x_3 \lor x_4) \land (x_2 \lor \neg x_4 \lor \neg x_5)\)

**Input**: a logic formula of size \(m\) containing \(n\) variables

**Output**: An assignment of Boolean values to the variables in the formula such that the formula is true

**Algorithm**: Try every variable assignment
Vertex Cover:

**Input**: A graph \((V,E)\) and a number \(m\)

**Output**: A subset \(S\) of \(V\) such that *for every edge* \((u,v)\) in \(E\), at least one of \(u\) or \(v\) is in \(S\) and \(|S|=m\) (if such an \(S\) exists)

**Algorithm**: Try every subset of vertices of size \(m\)
Traveling Salesman

Input: A complete weighted graph \((V,E)\) and a number \(m\)
Output: A circuit that visits each vertex exactly once and has total cost \(< m\) if one exists

Algorithm: Try every path, stop if find cheap enough one