CSE 332: Data Abstractions

Lecture 9: B Trees

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Announcements

- **Project 2** –
  Partner selection due by Fri *at the latest*.
- **Homework 3** – due next Wednesday

Today

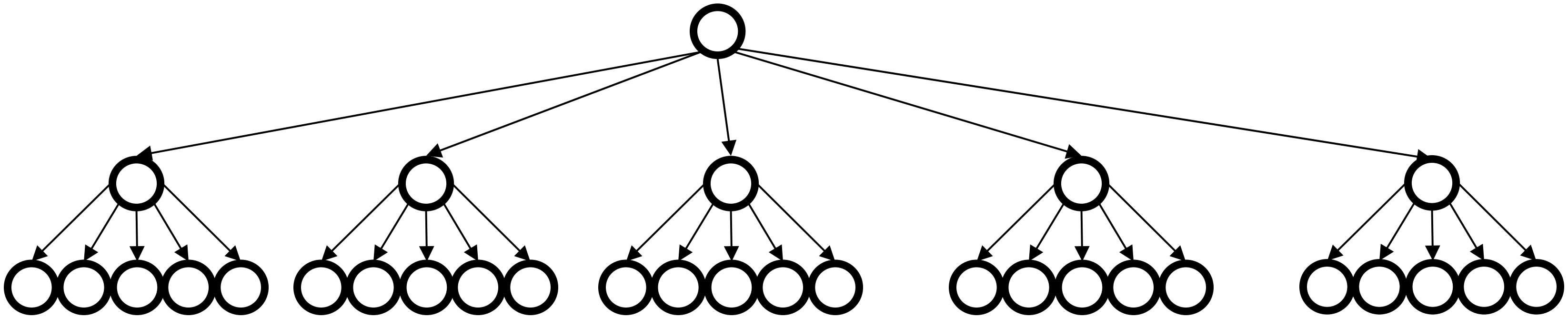
- Dictionaries
  - B-Trees

Our goal

- **Problem**: A dictionary with so much data most of it is on disk
- **Desire**: A balanced tree (logarithmic height) that is even shallower than AVL trees so that we can minimize disk accesses and exploit disk-block size
- **A key idea**: Increase the branching factor of our tree

M-ary Search Tree

- Build some sort of search tree with branching factor *M*:
  - Have an array of sorted children (`Node[]`)
  - Choose *M* to fit snugly into a disk block (1 access for array)

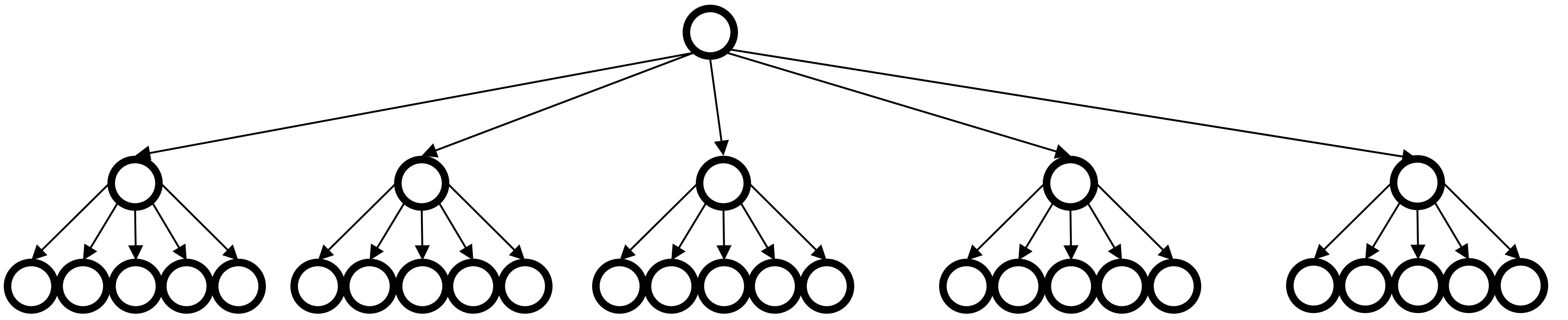


Perfect tree of height *h* has $(M^{h+1}-1)/(M-1)$ nodes (textbook, page 4)

What is the **height** of this tree?

What is the worst case running time of **find**?

M-ary Search Tree



- # hops for **`find?`**
  - If we have a balanced M-ary tree:
  - Approx. $\mathbf{log}_M\ n$ hops instead of $\mathbf{log}_2\ n$ (for balanced BST)
  - Example: M = 256 (=2^8) and n = 2^{40} that's 5 hops instead of 40 hops
- Sounds good, but how do we decide which branch to take?
  - Binary tree: Less than/greater than node value?
  - M-ary: In range 1? In range 2? In range 3?... In range M?
- Runtime of **`find`** if balanced: $O(\mathbf{log}_2\ M\ \mathbf{log}_M\ n)$
  - $\mathbf{log}_M$ n is the height we traverse.
  - $\log_2 M$: At each step, find the correct child branch to take using binary search among the M options!

Questions about M-ary search trees

- What should the **order** property be?
- How would you **rebalance** (ideally without more disk accesses)?
- Storing **real data** at inner-nodes (like we do in a BST) seems kind of wasteful…
  - To access the node, will have to load the **data** from disk, *even though most of the time we won't use it!!*
  - Usually we are just "passing through" a node on the way to the value we are actually looking for.

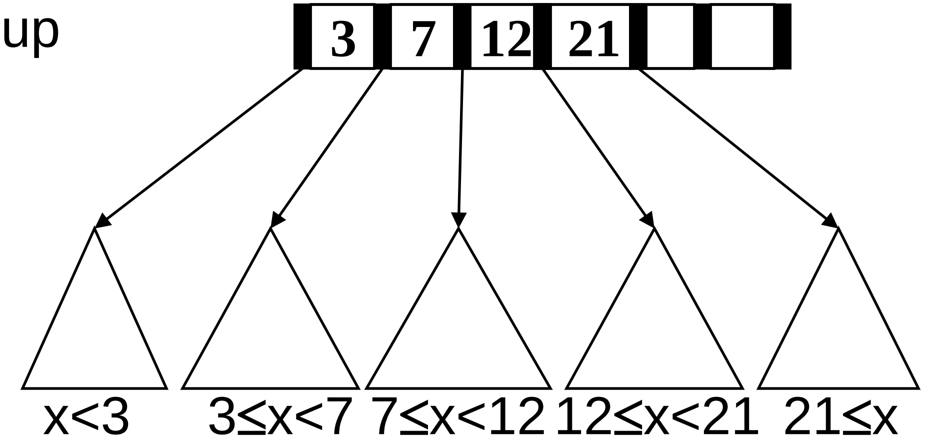
So let's use the branching-factor idea, but for a different kind of balanced tree:

- **Not** a binary *search tree*
- But still logarithmic height for any $M > 2$

B+ Trees (we and the book say "B Trees")

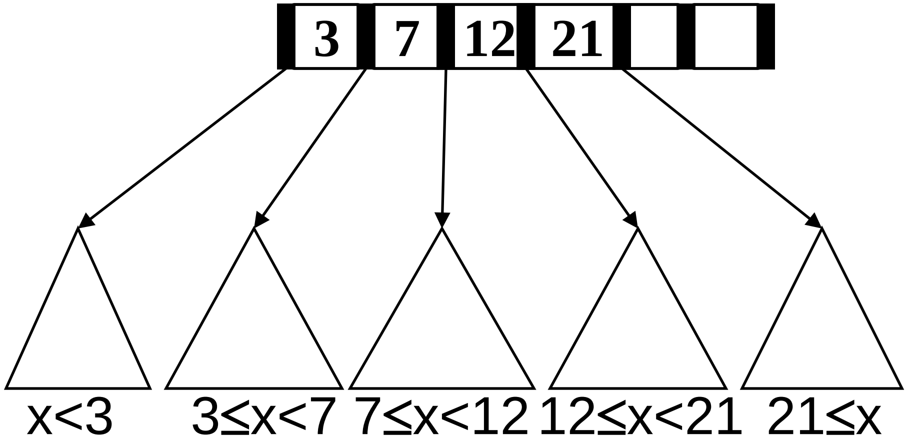
- Two types of nodes: **internal nodes** & **leaves**
- Each **internal node** has room for up to *M-1* keys and *M* children
  - No other data; all data at the leaves!
- Order property:

  Subtree **between** keys *a* and *b* contains only data that is ≥ ***a*** and < ***b*** (notice the ≥)
- **Leaf** nodes have up to *L* sorted data items
- As usual, we'll ignore the "along for the ride" data in our examples
  - Remember no data at non-leaves



Remember:
- **Leaves** store data
- **Internal nodes** are 'signposts'

Find



- Different from BST in that we ***don't store data at internal nodes***
- But **`find`** is still an easy root-to-leaf recursive algorithm
  - At each internal node do binary search on (up to) M-1 keys to find the branch to take
  - At the leaf do binary search on the (up to) L data items
- But to get logarithmic running time, we need a balance condition…

Structure Properties

- **Root** (special case)
  - If tree has ≤ *L* items, root is a leaf (occurs when starting up, otherwise unusual)
  - Else has between 2 and *M* children

- **Internal nodes**
  - Have between $\lceil M/2 \rceil$ and *M* children, i.e., at least half full

- **Leaf nodes**
  - All leaves at the same depth
  - Have between $\lceil L/2 \rceil$ and *L* data items, i.e., at least half full

Any *M* > 2 and *L* will work, but:

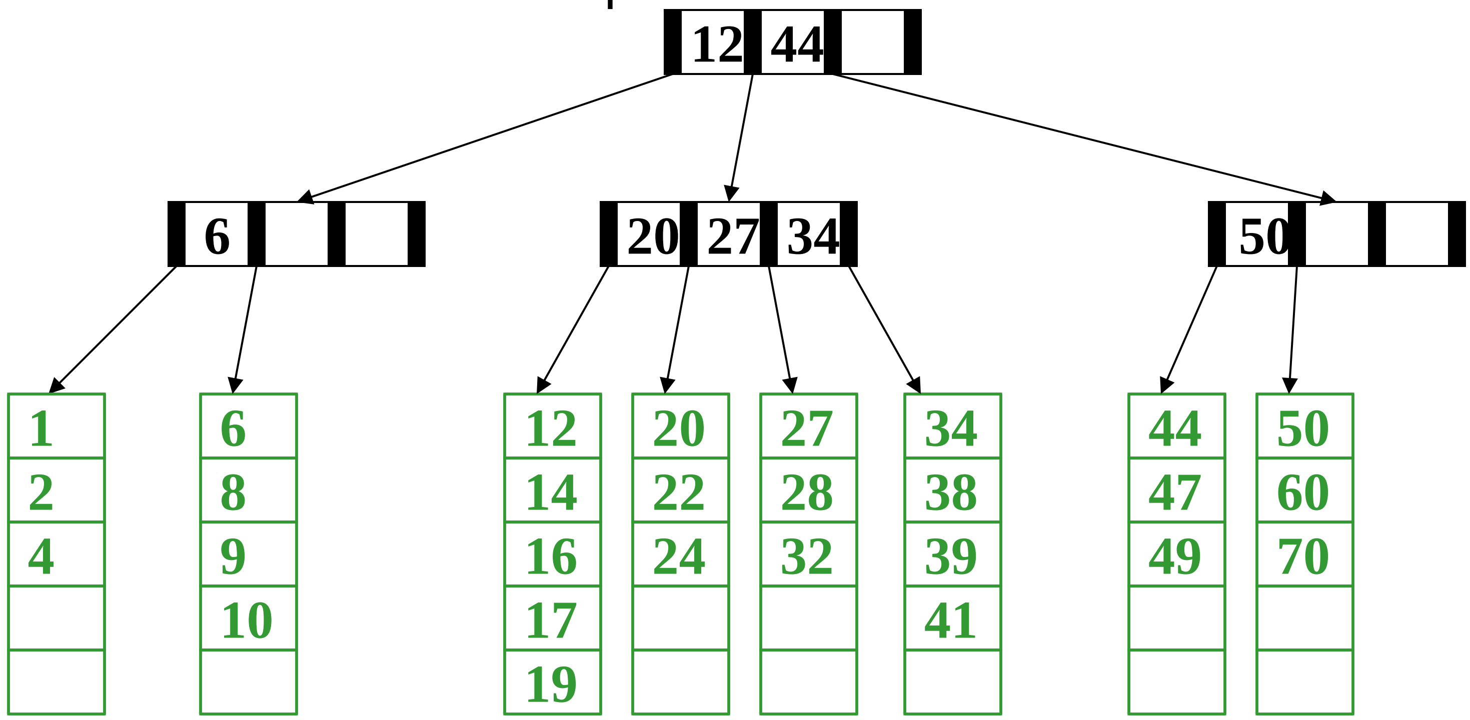
We pick M and L ***based on disk-block size***

Note on notation: Inner nodes drawn horizontally, leaves vertically to distinguish. Include empty cells

Example

Suppose *M*=4 (max # pointers in **internal node**)
and *L*=5 (max # data items at **leaf**)

- All **internal nodes** have at least 2 children
- All **leaves** have at least 3 data items (only showing keys)
- All **leaves** at same depth



Balanced enough

Not hard to show height *h* is logarithmic in number of data items *n*

- Let *M* > 2 (if *M* = 2, then a list tree is legal – no good!)
- Because all nodes are at least half full (except root may have only 2 children) and all leaves are at the same level, the minimum number of data items *n* for a height *h>0* tree is…

$$n \geq \underbrace{2 \lceil M/2 \rceil^{h-1}}_{\text{minimum number of leaves}} \underbrace{\lceil L/2 \rceil}_{\text{minimum data per leaf}}$$

Example: B-Tree vs. AVL Tree

Suppose we have 100,000,000 items

- Maximum height of AVL tree?

- Maximum height of B tree with M=128 and L=64?

Example: B-Tree vs. AVL Tree

Suppose we have 100,000,000 items

- **Maximum height of AVL tree**?
  - Recall S(h) = 1 + S(h-1) + S(h-2)
  - lecture7.xlsx reports: **37**

- **Maximum height of B tree** with *M*=128 and *L*=64?
  - Recall $(2\lceil M/2\rceil^{h-1})\lceil L/2\rceil$
  - lecture9.xlsx reports: **5** (and 4 is more likely)
  - Also not difficult to compute via algebra

Disk Friendliness

What makes B trees so disk friendly?

- Many keys stored in one **internal node**
  - All brought into memory in one disk access
    - *IF* we pick *M* wisely
  - Makes the binary search over *M*-1 keys totally worth it (insignificant compared to disk access times)

- **Internal nodes** contain only keys
  - Any **`find`** wants only one data item; wasteful to load unnecessary items with internal nodes
  - So only bring one **leaf** of data items into memory
  - Data-item size doesn't affect what *M* is

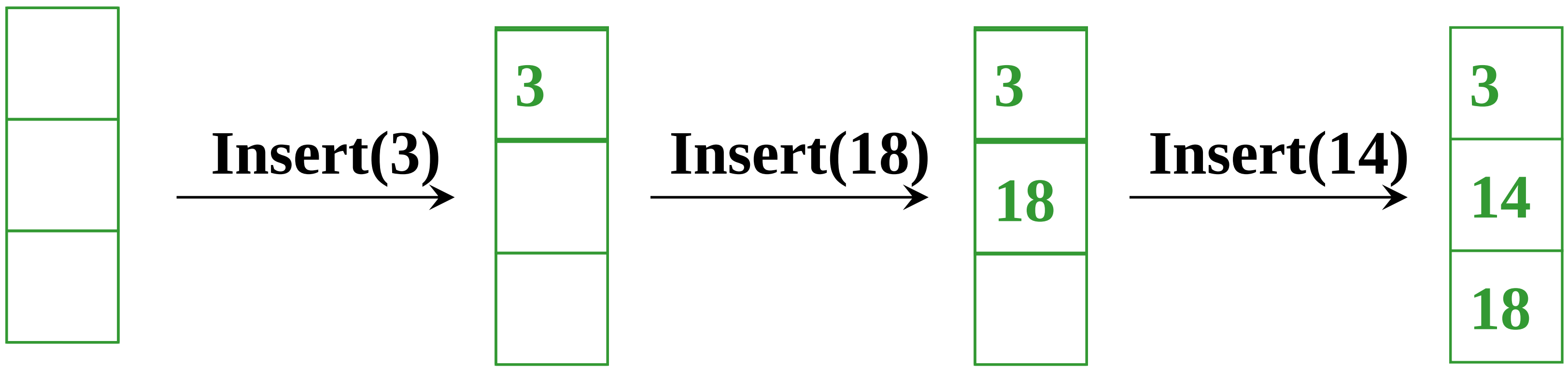
Maintaining balance

- So this seems like a great data structure (and it is)

- But we haven't implemented the other dictionary operations yet
  - **`insert`**
  - **`delete`**

- As with AVL trees, the hard part is maintaining structure properties
  - Example: for **`insert`**, there might not be room at the correct leaf

Building a B-Tree (insertions)

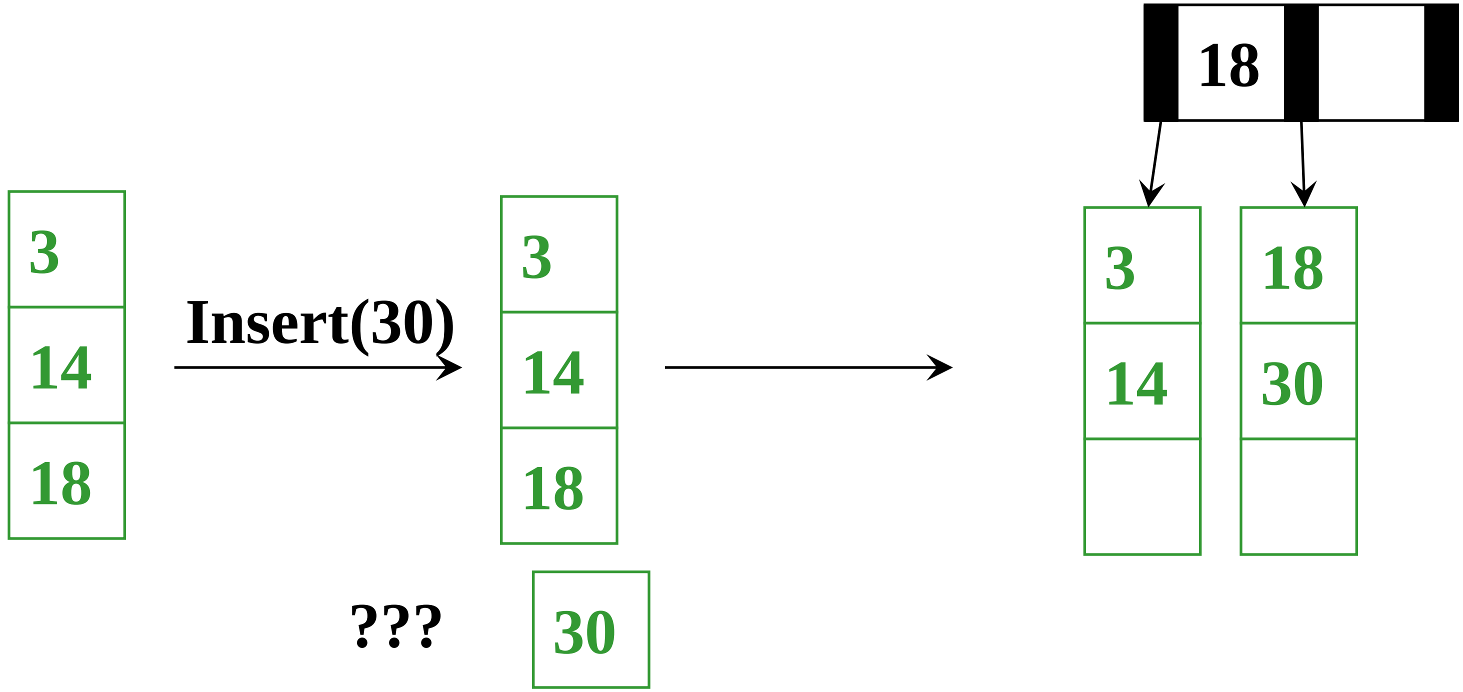


The empty B-Tree (the root will be a leaf at the beginning)

Just need to keep data in order

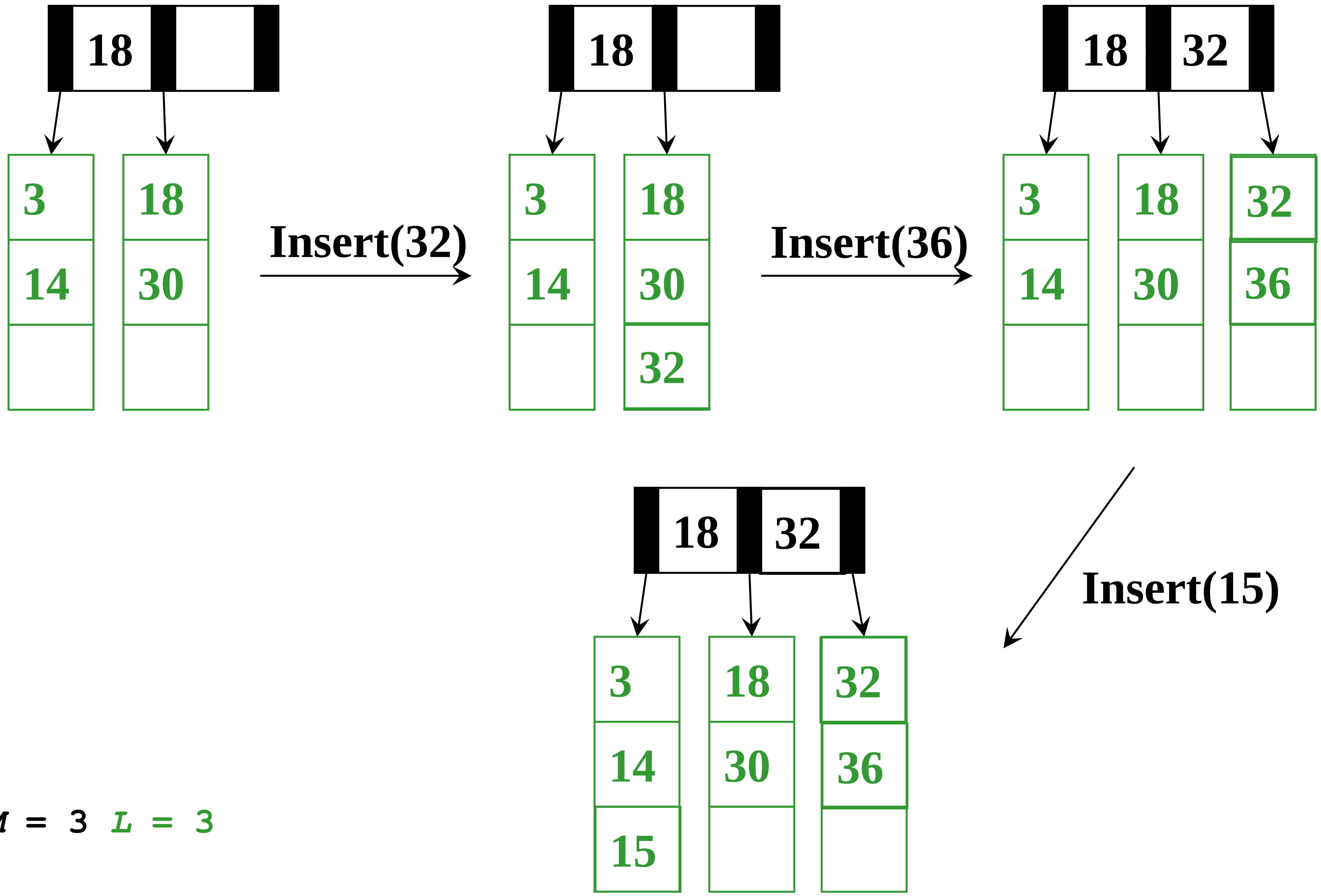
***M* = 3 *L* = 3**

M = 3 *L* = 3

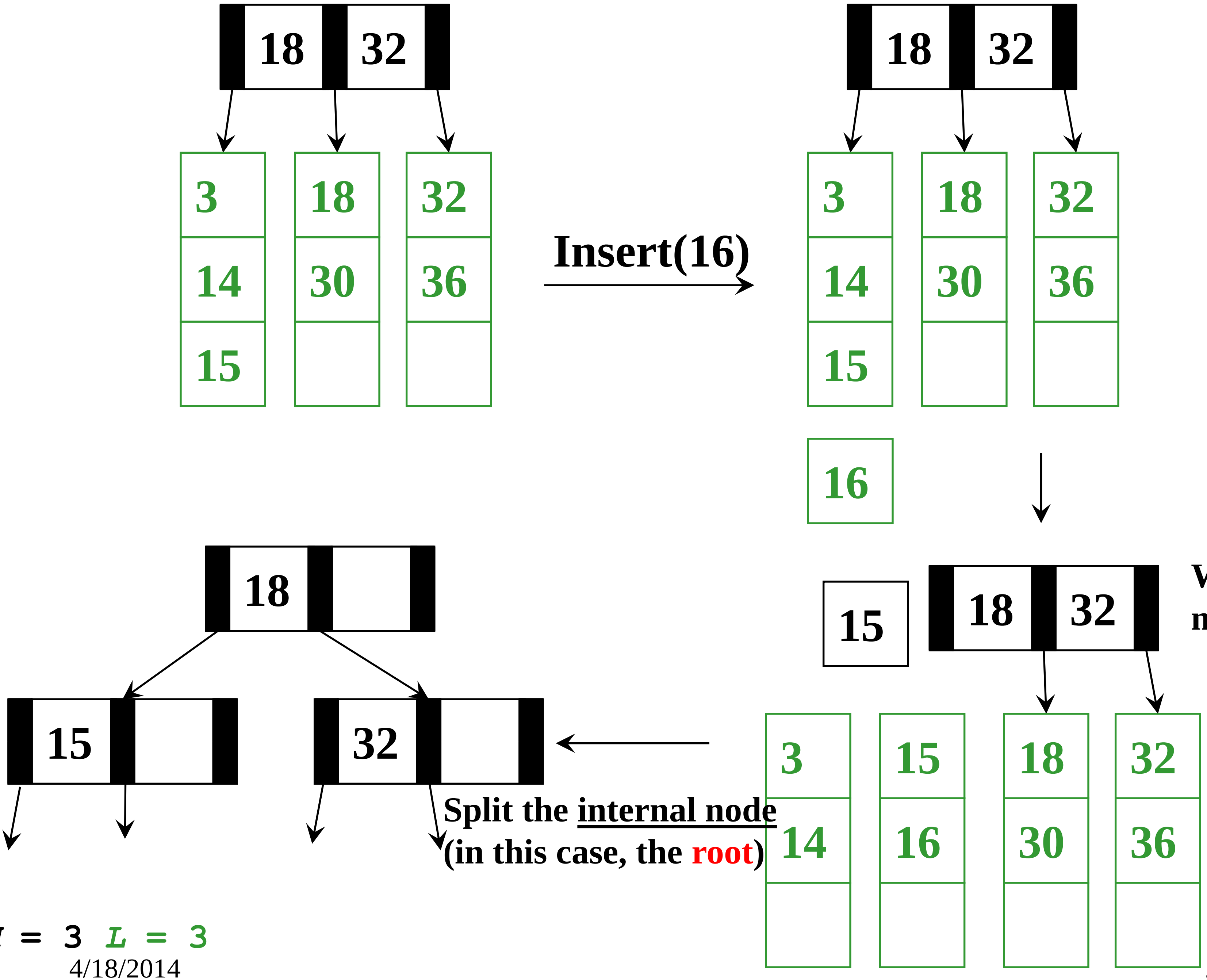


- When we ‘overflow’ a leaf, we split it into 2 leaves
- Parent gains another child
- If there is no parent (like here), we create one; how do we pick the key shown in it?
    - Smallest element in right tree

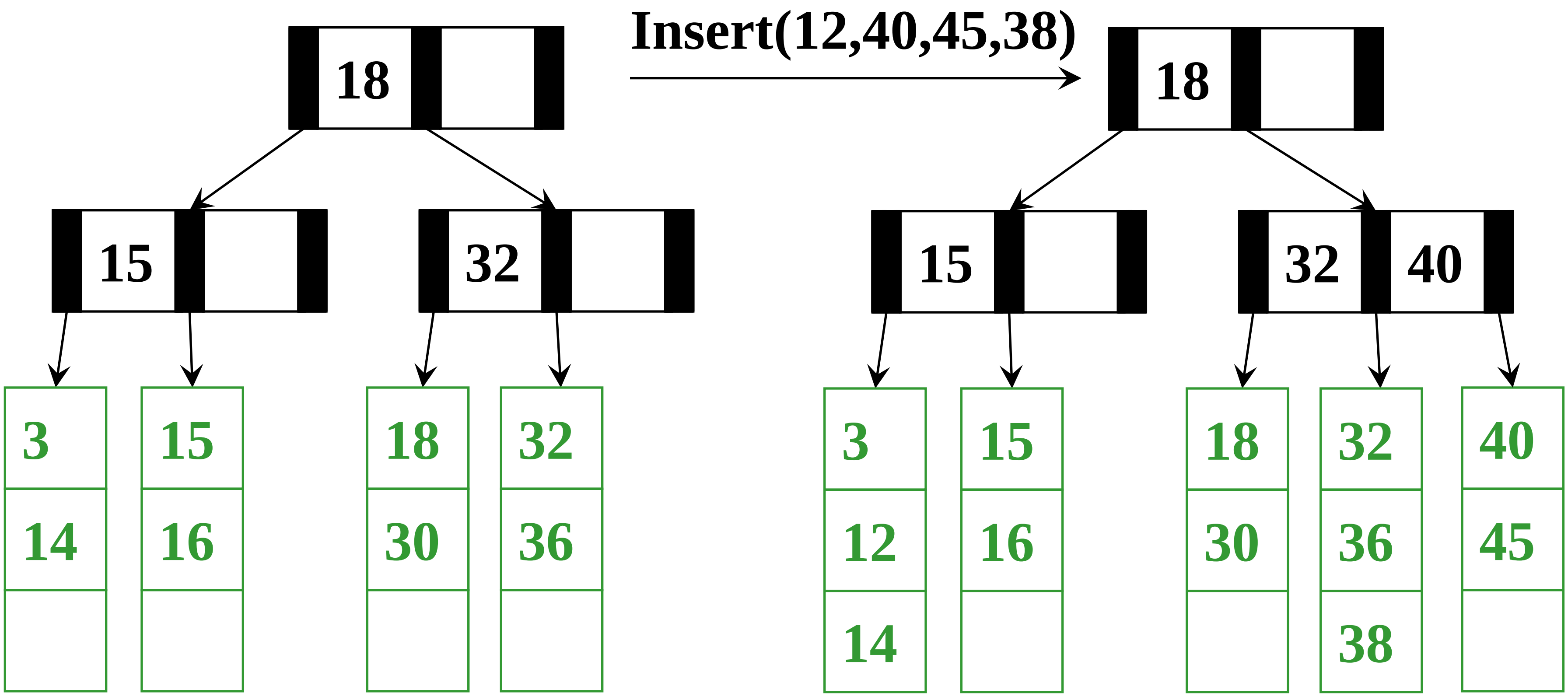
Split **leaf** again



M = 3 *L* = 3


18 32
3 14 15
18 30
32 36
Insert(16)
18 32
3 14 15
18 30
32 36
16
15
18 32
What now?
3 14
15 16
18 30
32 36
18
15
32
Split the internal node
(in this case, the root)

$M = 3$ $L = 3$



Note: Given the leaves and the structure of the tree, we can always fill in internal node keys;
'the smallest value in my right branch'

M **= 3** ***L*** **= 3**

Insertion Algorithm

1. Insert the data in its **leaf** in sorted order

2. If the **leaf** now has L+1 items, *overflow!*
   - Split the **leaf** into two nodes:
     - Original **leaf** with $\lceil (L+1)/2 \rceil$ smaller items
     - New **leaf** with $\lfloor (L+1)/2 \rfloor = \lceil L/2 \rceil$ larger items
   - Attach the new child to the parent
     - Adding new key to parent in sorted order

3. If step (2) caused the parent to have M+1 children, *overflow!*
   - …

Insertion algorithm continued

3. If an **internal node** has M+1 children
   - Split the **node** into **two nodes**
     - Original **node** with $\lceil (M+1)/2 \rceil$ smaller items
     - New **node** with $\lfloor (M+1)/2 \rfloor = \lceil M/2 \rceil$ larger items
   - Attach the new child to the parent
     - Adding new key to parent in sorted order

Splitting at a node (step 3) could make the parent overflow too

- *So repeat step 3 up the tree until a node doesn't overflow*
- If the **root** overflows, make a new **root** with two children
  - This is the only case that increases the tree height

Efficiency of insert

- Find correct leaf: $O(\mathbf{log_2}\, M\, \mathbf{log}_M\, n)$
- Insert in leaf: *O(L)*
- Split leaf: *O(L)*
- Split parents all the way up to root: $O(M\, \mathbf{log}_M\, n)$

Total: $O(L + M\, \mathbf{log}_M\, n)$

But it's not that bad:

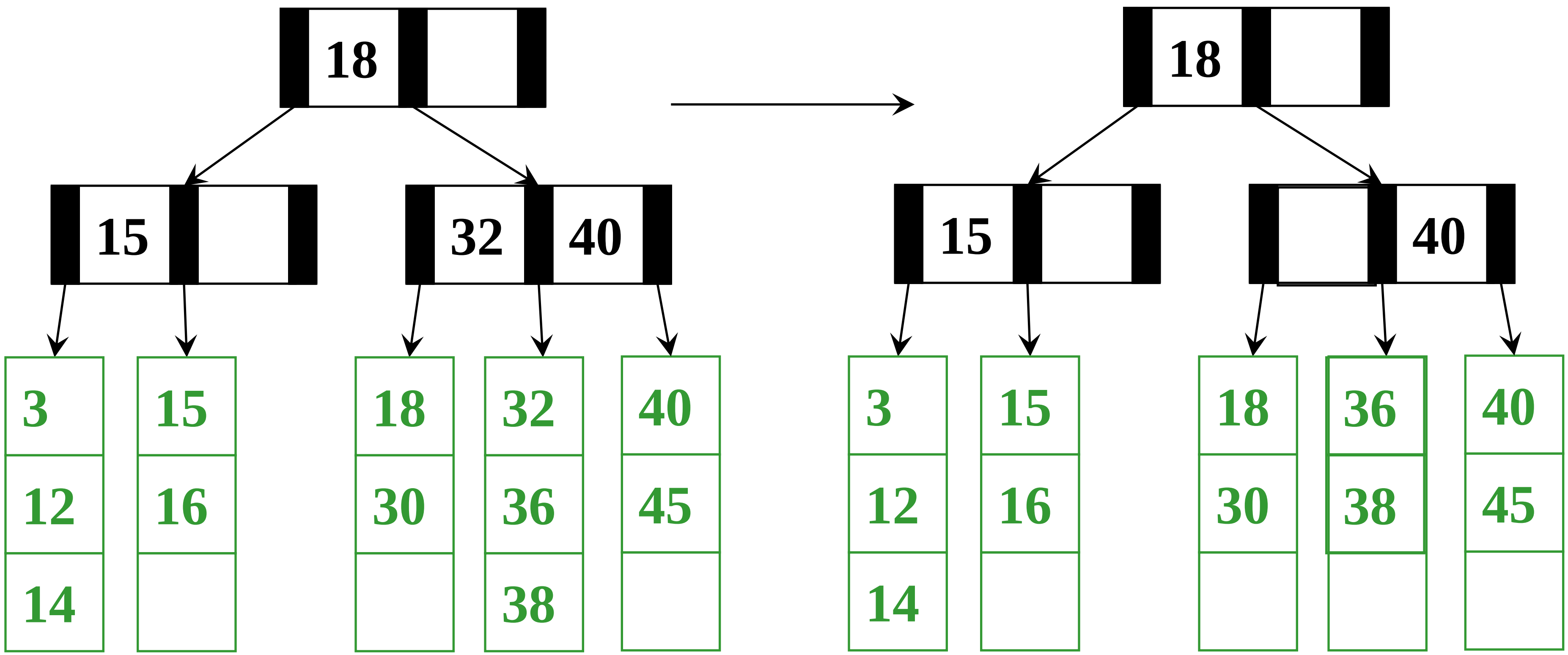
- Splits are not that common (only required when a node is FULL, M and L are likely to be large, and after a split, will be half empty)
- Splitting the **root** is extremely rare
- Remember disk accesses were the name of the game: $O(\mathbf{log}_M\, n)$

B-Tree Reminder: Another dictionary

- Before we talk about deletion, just keep in mind overall idea:
  - Large data sets won't fit entirely in memory
  - Disk access is slow
  - Set up tree so we do one disk access per node in tree
  - Then our goal is to keep tree shallow as possible
  - Balanced binary tree is a good start, but we can do better than $\log_2 n$ height
  - In an M-ary tree, height drops to $\log_M n$
    - Why not set M really really high? Height 1 tree…
    - Instead, set M so that each node fits in a disk block

And Now for Deletion…

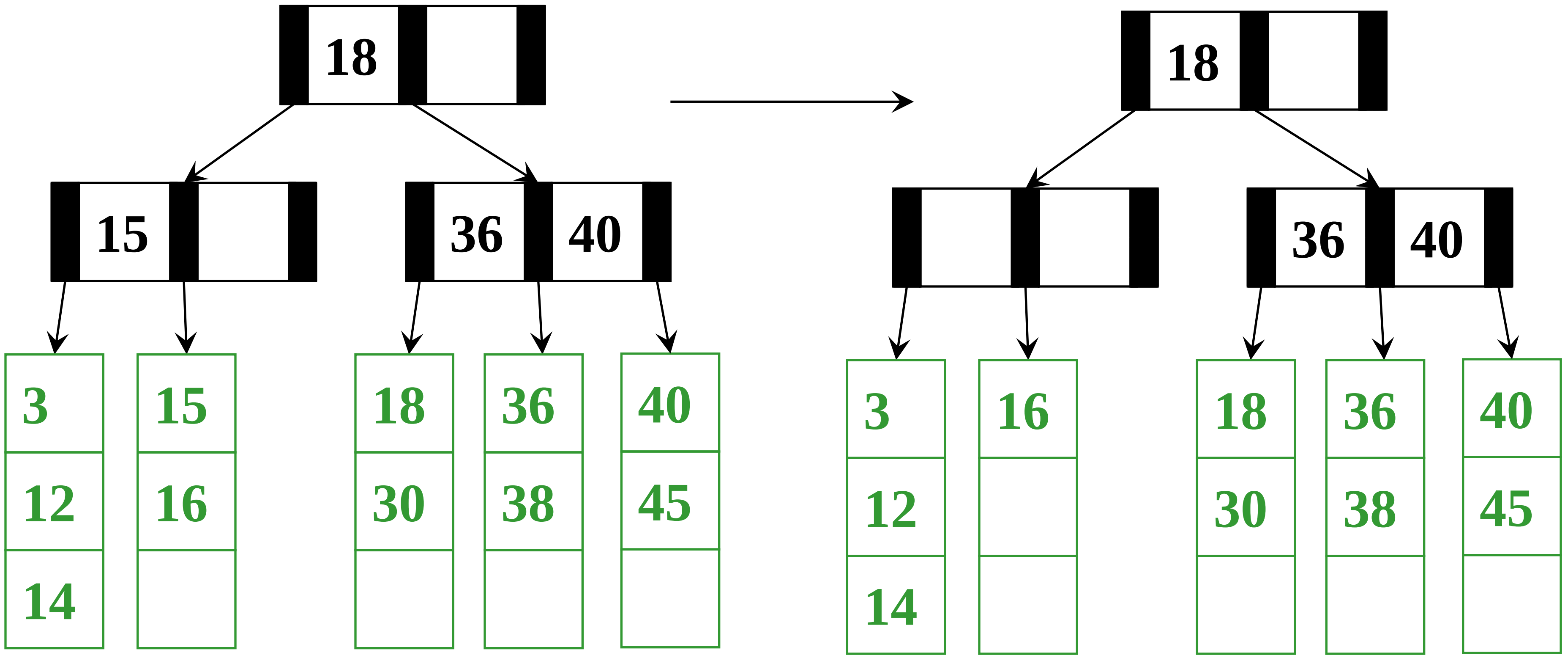
Delete(32)



Easy case: Leaf still has enough data; just remove

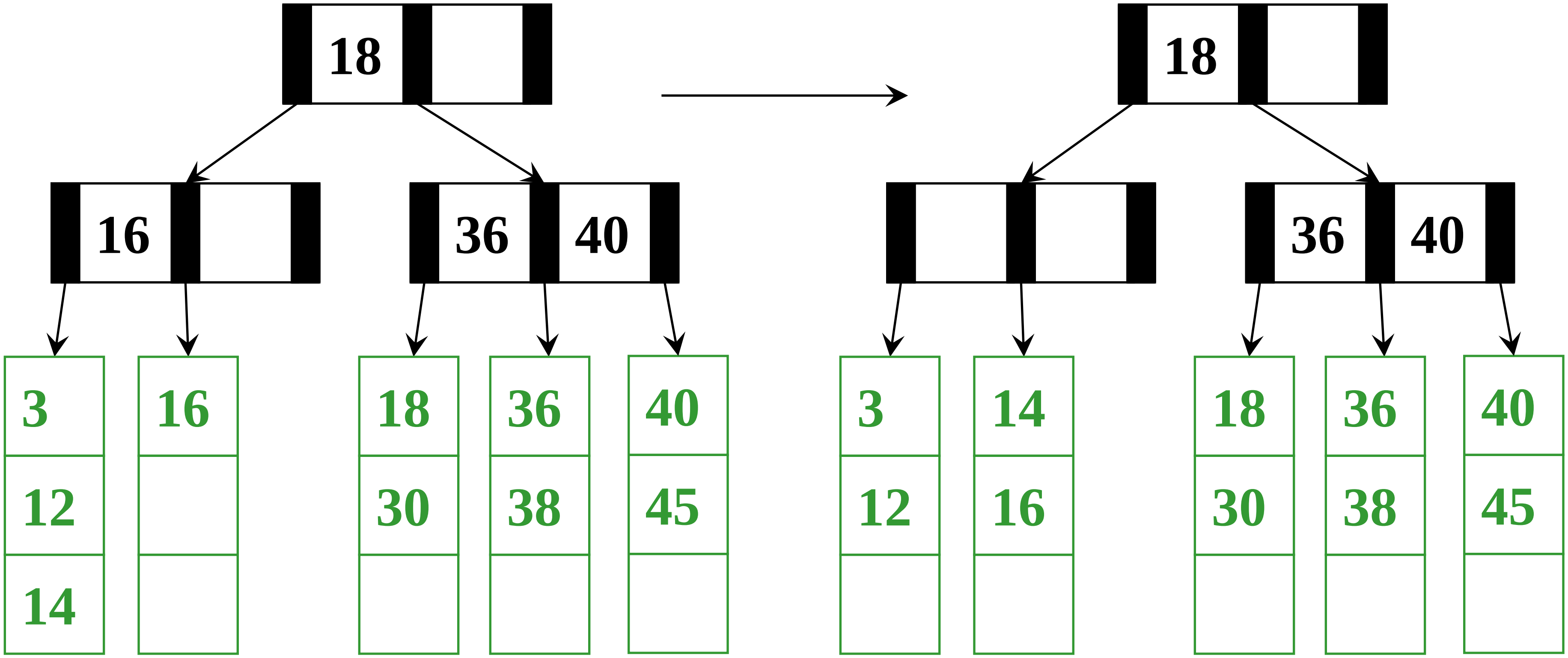
***M* = 3 *L* = 3**

Delete(15)



Is there a problem?

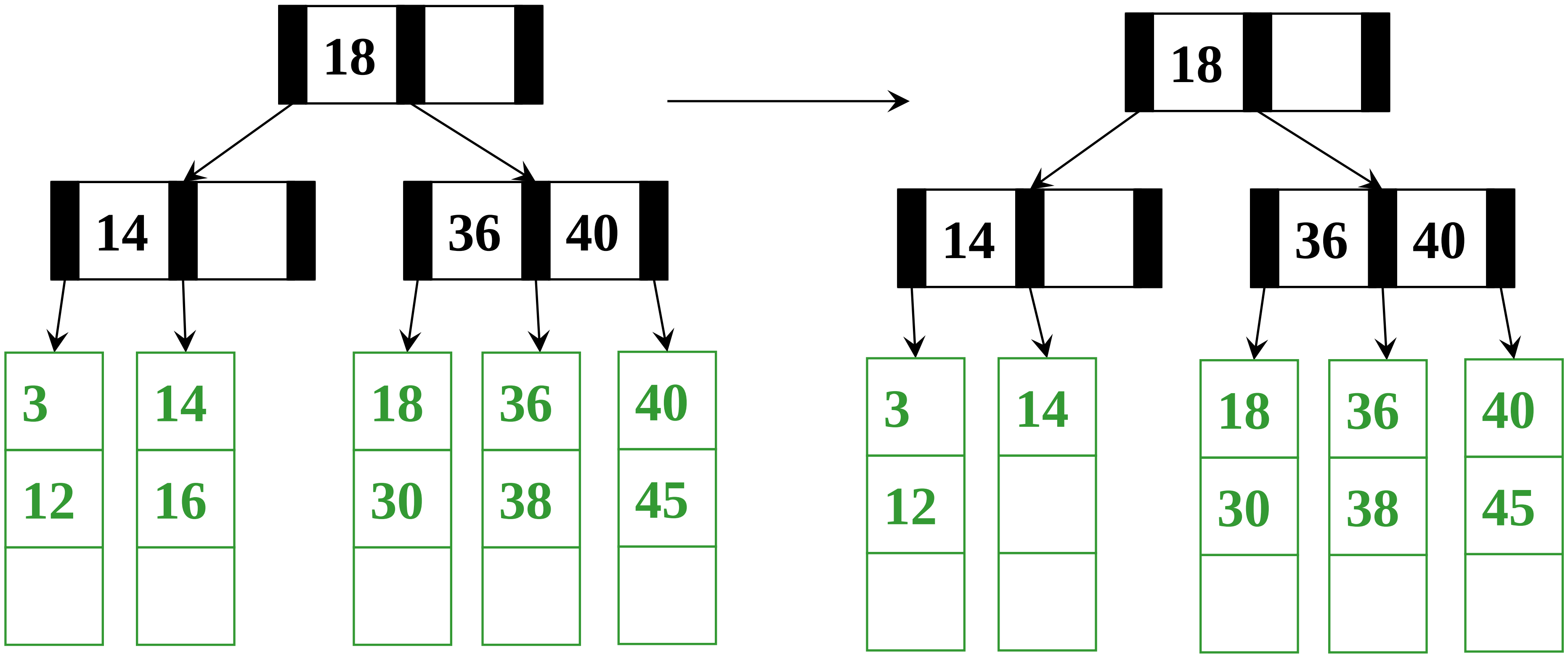
`M = 3` `L = 3`



$M = 3$ $L = 3$

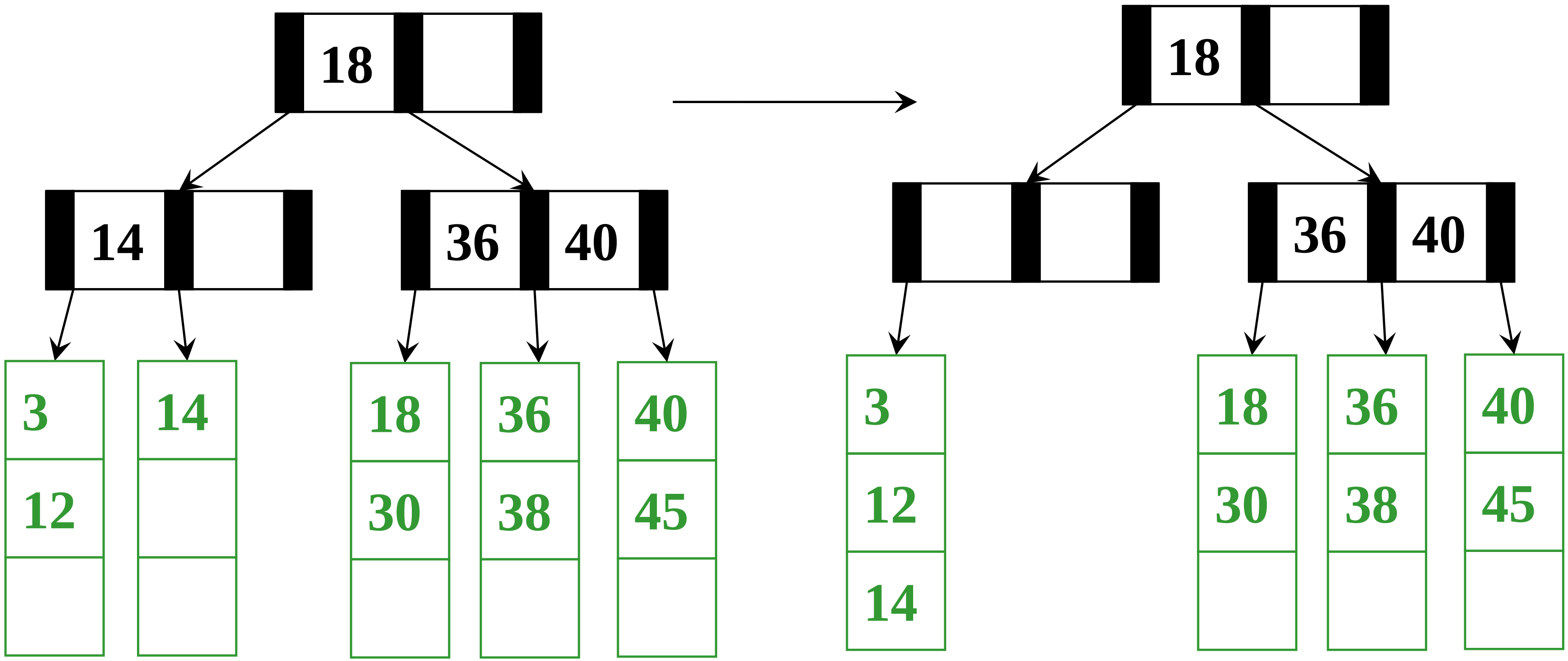
Adopt from neighbor!

Delete(16)



Is there a problem?

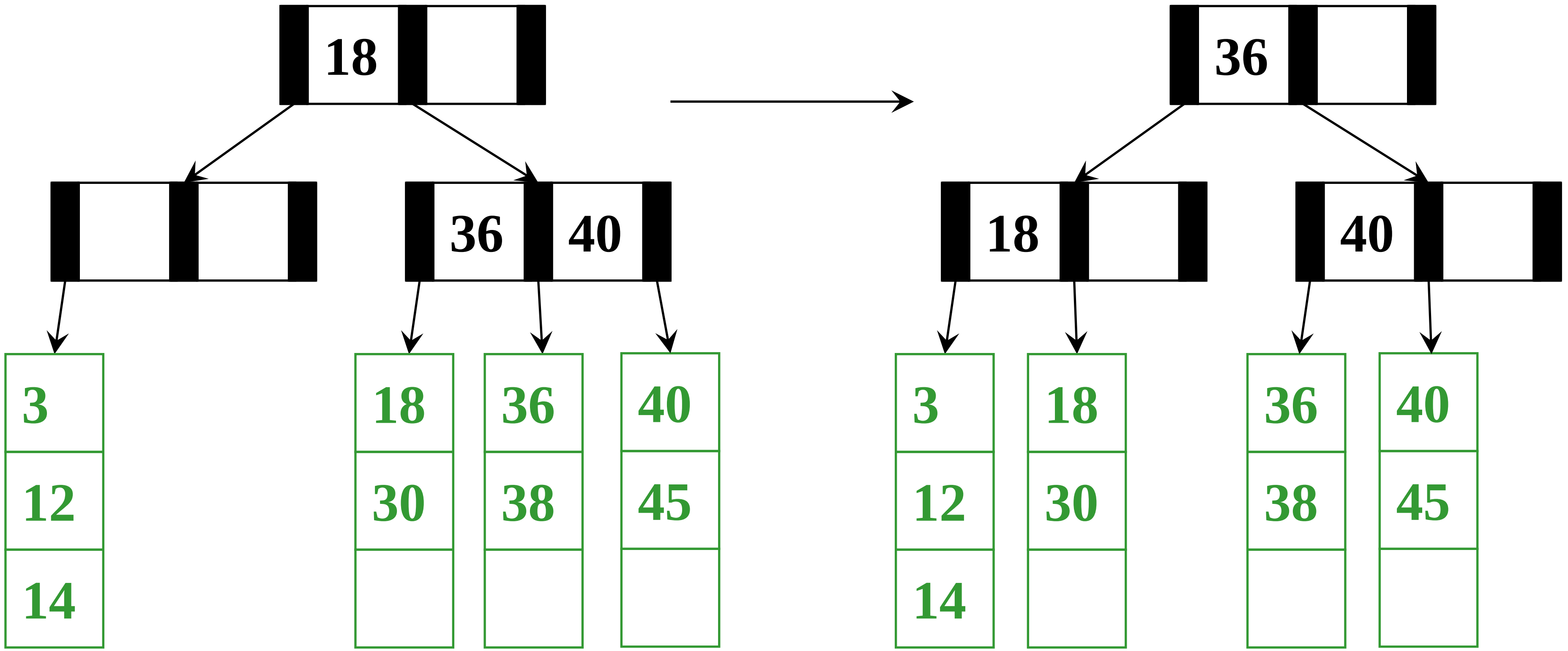
M = 3 *L* = 3



Merge with neighbor!

M = 3 **L = 3**

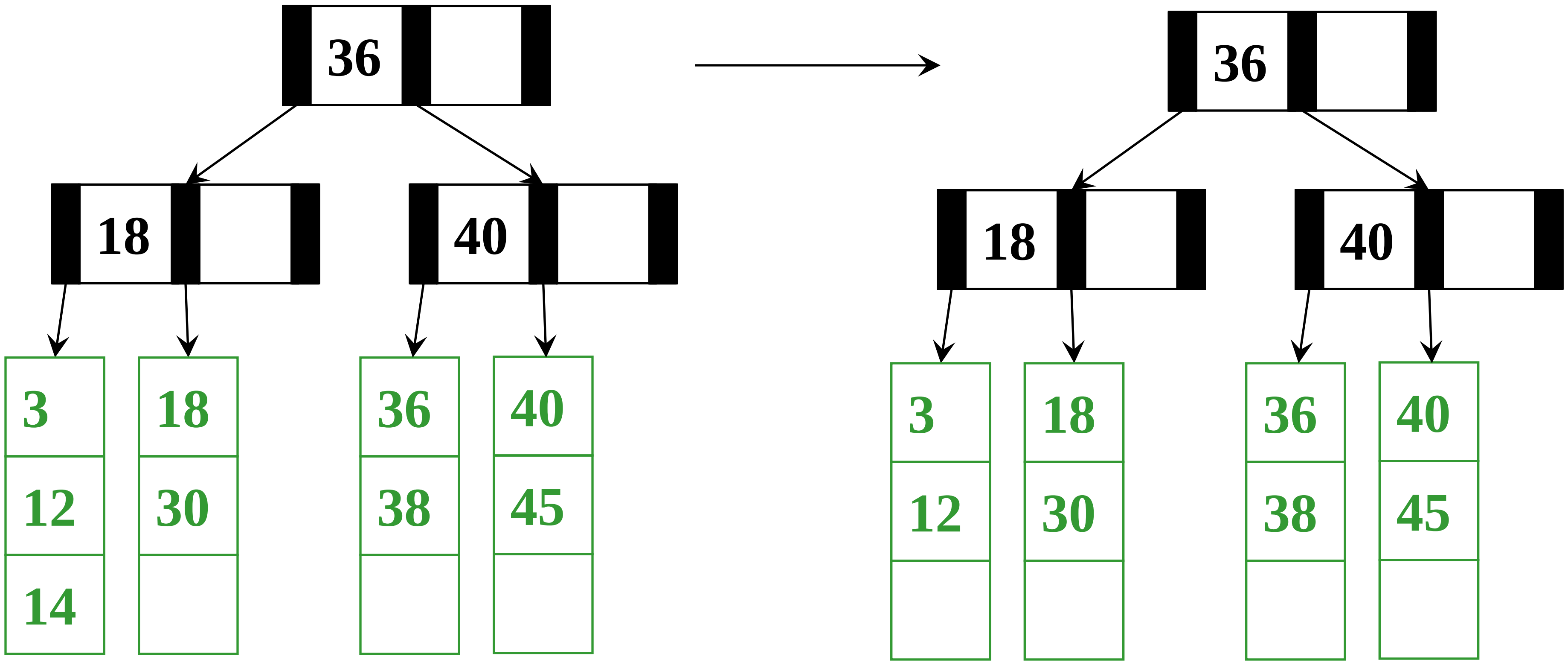
But hey, Is there a problem?



$M = 3$ $L = 3$

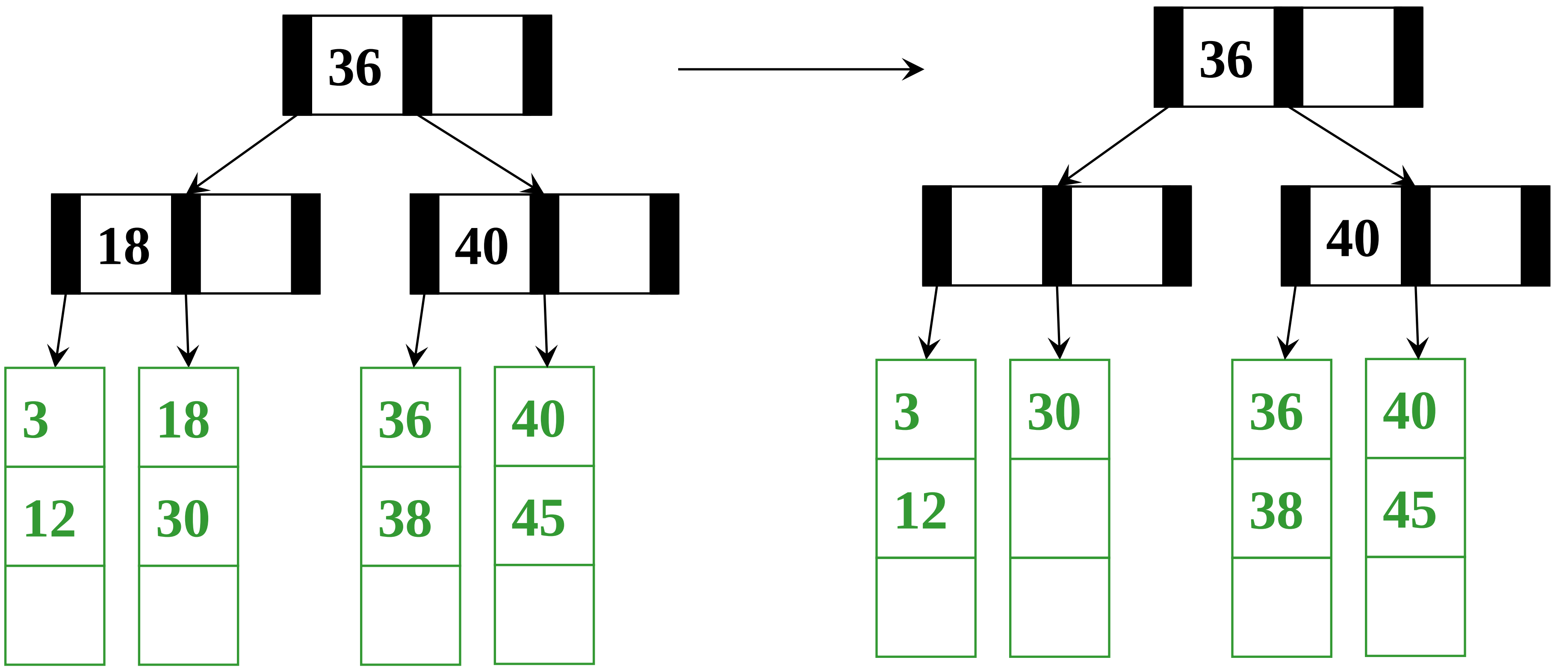
Adopt from neighbor!

Delete(14)



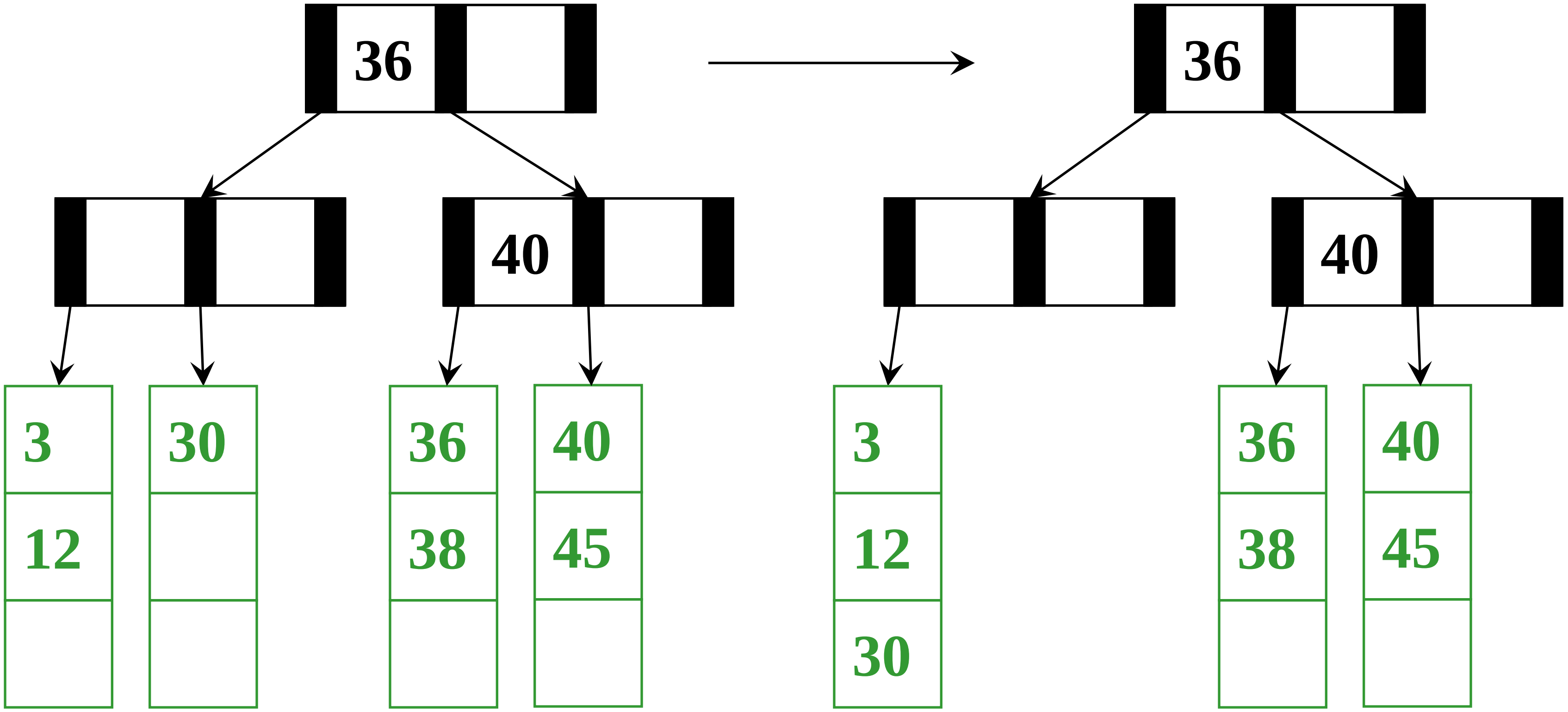
M = 3 L = 3

Delete(18)



Is there a problem?

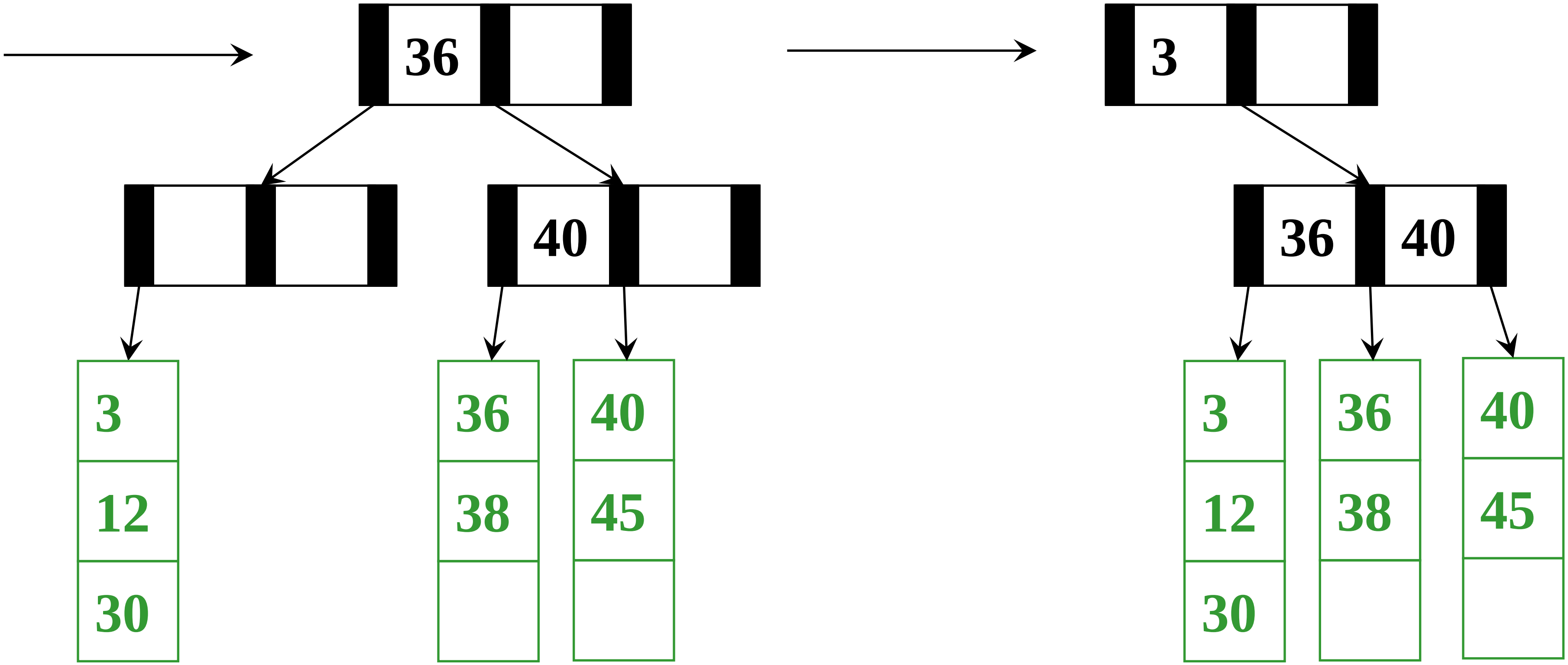
M = 3 *L* = 3



Merge with neighbor!

M = 3 L = 3

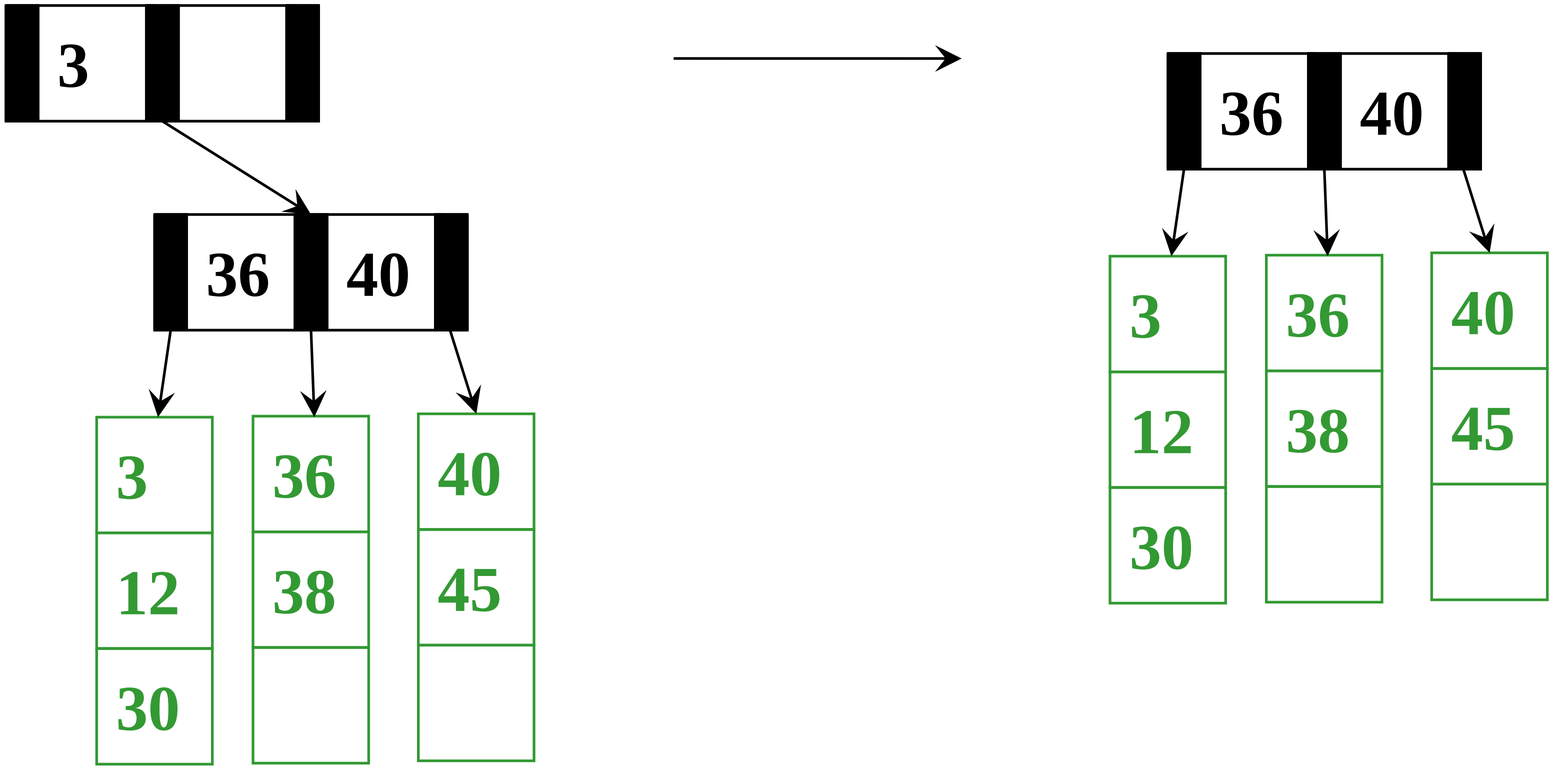
But hey, Is there a problem?



M = 3 *L* = 3

Merge with neighbor!

But hey, Is there a problem?



Pull out the root!

***M* = 3** ***L* = 3**

Deletion Algorithm, part 1

1. Remove the data from its leaf

2. If the leaf now has $\lceil \mathbf{L/2} \rceil$ **- 1**, *underflow!*
   – If a neighbor has > $\lceil \mathbf{L/2} \rceil$ items, *adopt* and update parent
   – Else *merge* node with neighbor
     • Guaranteed to have a legal number of items
     • Parent now has one less node

3. If step (2) caused the parent to have $\lceil \mathbf{M/2} \rceil$ **- 1** children, *underflow!*
   – …

Deletion algorithm (continued)

3. If an internal node has $\lceil M/2 \rceil - 1$ children
   - If a neighbor has > $\lceil M/2 \rceil$ items, *adopt* and update parent
   - Else *merge* node with neighbor
     - Guaranteed to have a legal number of items
     - Parent now has one less node, may need to continue up the tree

If we merge all the way up through the root, that's fine unless the root went from 2 children to 1

- In that case, delete the root and make child the root
- This is the only case that decreases tree height

Worst-Case Efficiency of Delete

- Find correct leaf: $O(\log_2 M \log_M n)$
- Remove from leaf: $O(L)$
- Adopt from or merge with neighbor: $O(L)$
- Adopt or merge all the way up to root: $O(M \log_M n)$

Total: $O(L + M \log_M n)$

But it's not that bad:

- Merges are not that common
- Disk accesses are the name of the game: $O(\log_M n)$

Insert vs delete comparison

Insert

- Find correct leaf: $O(\log_2 M \log_M n)$
- Insert in leaf: $O(L)$
- Split leaf: $O(L)$
- Split parents all the way up to root: $O(M \log_M n)$

Delete

- Find correct leaf: $O(\log_2 M \log_M n)$
- Remove from leaf: $O(L)$
- Adopt/merge from/with neighbor leaf: $O(L)$
- Adopt or merge all the way up to root: $O(M \log_M n)$

B Trees in Java?

For most of our data structures, we have encouraged writing high-level, reusable code, such as in Java with generics

It is worthwhile to know enough about "how Java works" to understand why this is probably a bad idea for B trees

- If you just want a balanced tree with worst-case logarithmic operations, no problem
  - If M=3, this is called a 2-3 tree
  - If M=4, this is called a 2-3-4 tree
- Assuming our goal is efficient number of disk accesses
  - Java has many advantages, but it wasn't designed for this

The key issue is extra *levels of indirection*…

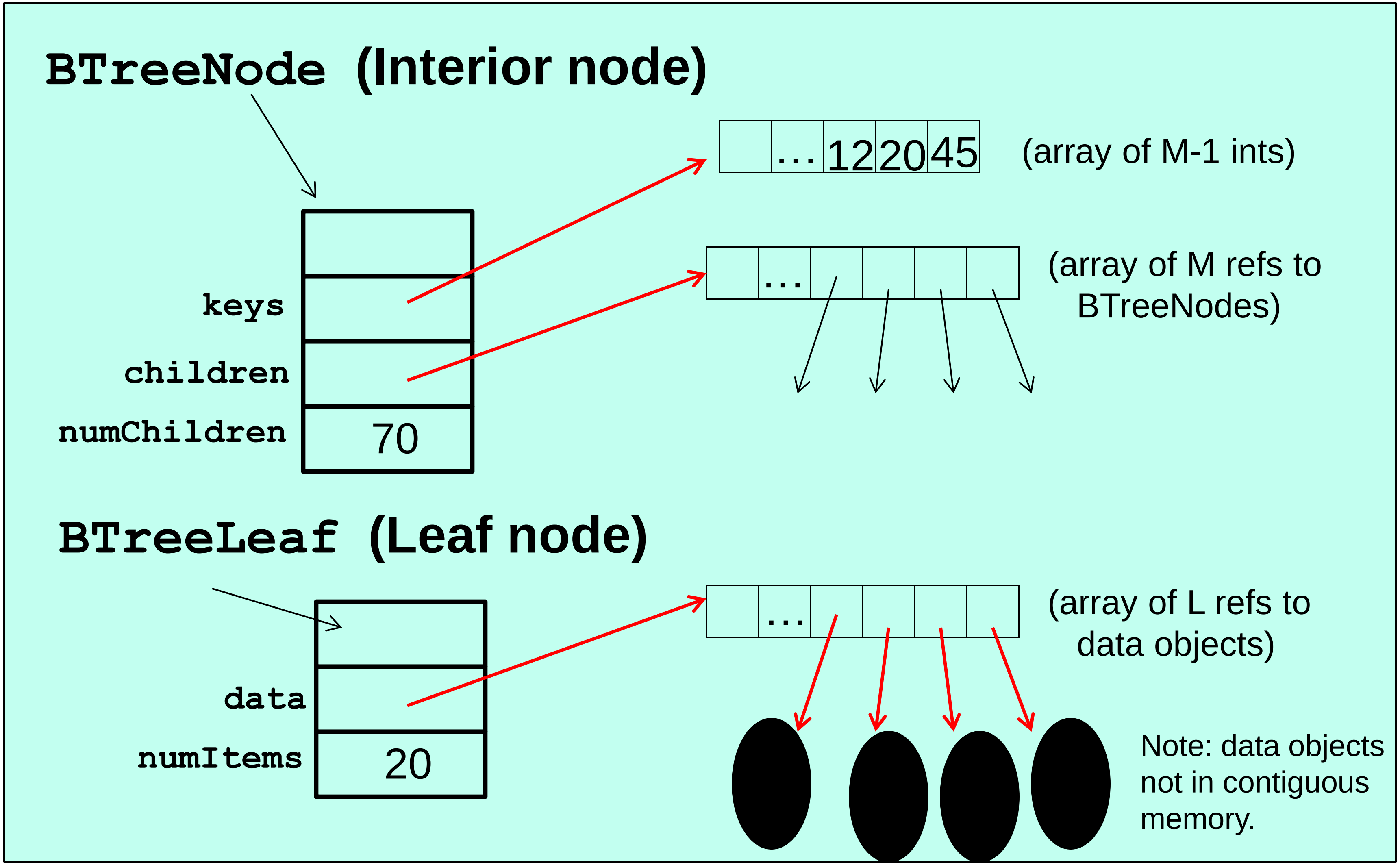
Naïve approach in Java

Even if we assume data items have **`int`** keys, you cannot get the data representation you want for "really big data"

```
interface Keyed {
  int getKey();
}
class BTreeNode<E implements Keyed> {
  static final int M = 128;
  int[]          keys        = new int[M-1];
  BTreeNode<E>[] children    = new BTreeNode[M];
  int            numChildren = 0;
  …
}
class BTreeLeaf<E implements Keyed> {
  static final int L = 32;
  E[] data      = (E[])new Object[L];
  int numItems = 0;
  …
}
```

What that looks like in Java

All the red references indicate "unnecessary" indirection that might be avoided in another programming language.



The moral

- The whole idea behind B trees was to keep related data in contiguous memory

- But that's "the best you can do" in Java
  - Again, the advantage is generic, reusable code
  - But for your performance-critical web-index, not the way to implement your B-Tree for terabytes of data

- Other languages (e.g., C++) have better support for "flattening objects into arrays"

- Levels of indirection matter!

Conclusion: Balanced Trees

- *Balanced* trees make good dictionaries because they guarantee logarithmic-time **`find`**, **`insert`**, and **`delete`**
  - Essential and beautiful computer science
  - But only if you can maintain balance within the time bound
- AVL trees maintain balance by tracking height and allowing all children to differ in height by at most 1
- B trees maintain balance by keeping nodes at least half full and all leaves at same height
- Other great balanced trees (see text; worth knowing they exist)
  - Red-black trees: all leaves have depth within a factor of 2
  - Splay trees: self-adjusting; amortized guarantee; no extra space for height information