CSE332: Data Abstractions

Lecture 2: Algorithm Analysis

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Announcements

- Project 1 – phase A due Monday
- Homework 1 – (out today) due next Friday (normally due on Wed)
- Office Hours – posted soon

Today

- Finish discussing queues

- Begin analyzing algorithms
  - Using asymptotic analysis (continue next time)

Algorithm Analysis

- Correctness:
  - Does the algorithm do what is intended.
- Performance:
  - Speed time complexity
  - Memory space complexity
- Why analyze?
  - To make good design decisions
  - Enable you to look at an algorithm (or code) and identify the bottlenecks, etc.

Correctness

Correctness of an algorithm is established by proof. Common approaches:

- (Dis)proof by counterexample
- Proof by contradiction
- Proof by induction
  - Especially useful in recursive algorithms

Proof by Induction

- **Base Case**: The algorithm is correct for a base case or two by inspection.
- **Inductive Hypothesis (n=k):** Assume that the algorithm works correctly for the first k cases.
- **Inductive Step (n=k+1):** Given the hypothesis above, show that the k+1 case will be calculated correctly.

Mathematical induction

Suppose $P(n)$ is some predicate (involving integer n)

- Example: $n \geq n/2 + 1$ (for all $n \geq 2$)

To prove $P(n)$ for all integers $n \geq c$, it suffices to prove

1. $P(c)$ – called the "basis" or "base case"
2. If $P(k)$ then $P(k+1)$ – called the "induction step" or "inductive case"

We will use induction:

To show an algorithm is correct or has a certain running time *no matter how big a data structure or input value is*

(Our "*n*" will be the data structure or input size.)

$P(n)$ = " the sum of the first n powers of 2 (starting at 2^0) is 2^n-1 "

Inductive Proof Example

Theorem: $P(n)$ holds for all $n \geq 1$

Proof: By induction on n

- Base case, n=1: Sum of first power of 2 is 2^0, which equals 1.
  And for n=1, 2^n-1 equals 1.
- Inductive case:
  - Inductive hypothesis: Assume the sum of the first k powers of 2 is 2^k-1
  - Show, given the hypothesis, that the sum of the first (k+1) powers of 2 is 2^{k+1}-1

From our inductive hypothesis we know:

$$1+2+4+...+2^{k-1}=2^k-1$$

Add the next power of 2 to both sides…

$$1+2+4+...+2^{k-1}+2^k=2^k-1+2^k$$

We have what we want on the left; massage the right a bit

$$1+2+4+...+2^{k-1}+2^k=2(2^k)-1=2^{k+1}-1$$

Note for homework

Proofs by induction will come up a fair amount on the homework

When doing them, be sure to state each part clearly:

- What you’re trying to prove
- The base case
- The inductive case
- The inductive hypothesis
  - In many inductive proofs, you’ll prove the inductive case by just starting with your inductive hypothesis, and playing with it a bit, as shown above

How should we compare two algorithms?

Gauging performance

- Uh, why not just run the program and time it
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation (“coding it up”)

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is "plenty good" for small inputs (if *n* is 10, probably anything is fast enough)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

- Can do analysis before coding!

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Conditionals	Time of condition plus time of slower branch
Loops	Num iterations * time for loop body
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Complexity cases

We’ll start by focusing on two cases:

- **Worst-case complexity**: max # steps algorithm takes on “most challenging” input of size N
- **Best-case complexity**: min # steps algorithm takes on “easiest” input of size N

Example

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case:

Worst case:

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6 “ish” steps = $O(1)$

Worst case: 5 “ish” * (arr.length)

= O(arr.length)

Analyzing Recursive Code

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
  - Conceptually, in each recursive call we:
    - Perform some amount of work, call it w(n)
    - Call the function recursively with a smaller portion of the list

- So, if we do w(n) work per step, and reduce the problem size in the next recursive call by 1, we do total work:

$$T(n)=w(n)+T(n-1)$$

- With some base case, like T(1)=5=O(1)

Example Recursive code: sum array

Recursive:

- Recurrence is some constant amount of work O(1) done *n* times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

Each time **help** is called, it does that O(1) amount of work, and then calls **help** again on a problem one less than previous problem size.

Recurrence Relation: $T(n) = O(1) + T(n\text{-}1)$

Solving Recurrence Relations

- Say we have the following recurrence relation:

  T(n)=6 “ish”+T(n-1)

  T(1)=9 “ish” ←base case

- Now we just need to solve it; that is, reduce it to a closed form.
- Start by writing it out:

$$
\begin{aligned}
T(n)&=6+T(n-1)\\
&=6+6+T(n-2)\\
&=6+6+6+T(n-3)\\
&=6k+T(n-k)\\
&=6+6+6+\ldots+6+T(1) = 6+6+6+\ldots+6+9\\
&=6k+9, \text{ where k is the \# of times we expanded T()}
\end{aligned}
$$

- We expanded it out n-1 times, so

$$
\begin{aligned}
T(n)&=6k+T(n-k)\\
&=6(n-1)+T(1) = 6(n-1)+9\\
&=6n+3 = O(n)
\end{aligned}
$$

Or When does n-k=1?
Answer: when k=n-1

Binary search

Best case:

Worst case:

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

– Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Binary search

Best case: 9 "ish" steps = $O(1)$

Worst case: $T(n) = 10$ "ish" + $T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

Solving Recurrence Relations

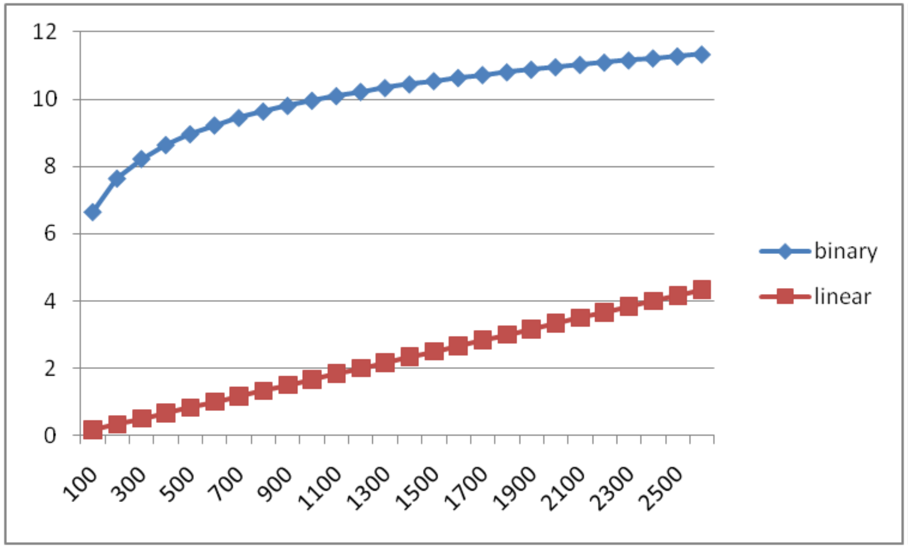
1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $\quad$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   – $T(n) = 10 + 10 + T(n/4)$
   $= 10 + 10 + 10 + T(n/8)$
   $= \ldots$
   $= 10k + T(n/(2^k))$ (where k is the number of expansions)
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   – $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   – So $T(n) = 10 \log_2 n + 15$ (get to base case and do it)
   – So $T(n)$ is $O(\log n)$

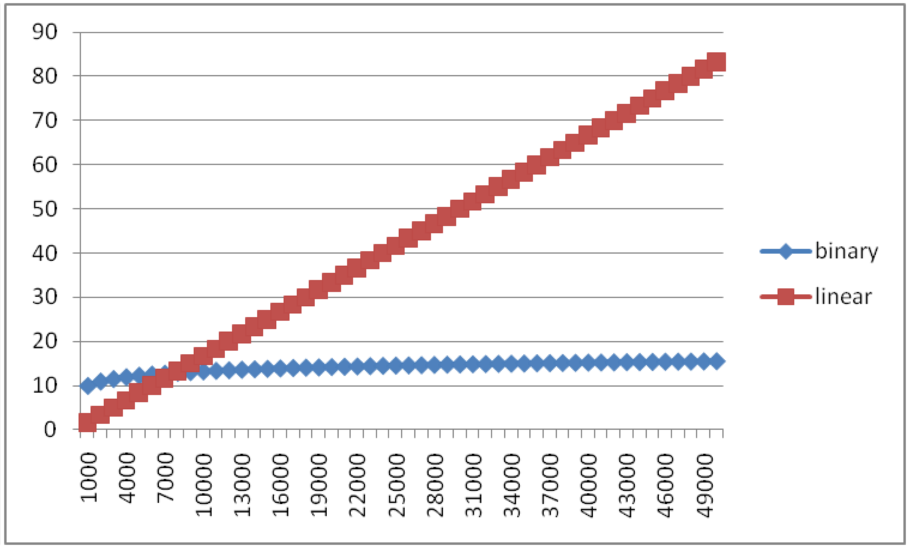
Ignoring constant factors

- So binary search is $O(\mathtt{log}\ n)$ and linear is $O(n)$
  - But which will actually be faster?
  - Depending on constant factors and size of n, in a particular situation, linear search could be faster….
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each n
  - And could depend on size of n
- **But** there exists some n_0 such that for all $n > n_0$ binary search wins
- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!





Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

– Recurrence is $c + c + \ldots + c$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a binary version of sum?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \ldots$ for **`log`** n times
- $2^{(\log n)} - 1$ which is proportional to n (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than $O(n)$ – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed, the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O($`log` $n)$: same recurrence as for `find`

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	O(n `log` n)

Note big-Oh can also use more than one variable

- Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$