CSE 332: Data Abstractions

Lecture 3: Asymptotic Analysis

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Autumn 2013
Announcements

• **Project 1** – phase A due Wed
• **Homework 1** – due Friday at *beginning* of class

• Any more Info sheets?
Today

• How to compare two algorithms?
• Analyzing code
• Big-Oh
Comparing Two Algorithms…
Gauging performance

• Uh, why not just run the program and time it
  – Too much \textit{variability}, not reliable or \textit{portable}:
    • Hardware: processor(s), memory, etc.
    • OS, Java version, libraries, drivers
    • Other programs running
    • Implementation dependent
  – Choice of input
    • Testing (inexhaustive) may \textit{miss} worst-case input
    • Timing does not \textit{explain} relative timing among inputs
      (what happens when $n$ doubles in size)

• Often want to evaluate an \textit{algorithm}, not an implementation
  – Even \textit{before} creating the implementation (“coding it up”)
Comparing algorithms

When is one algorithm (not implementation) better than another?
- Various possible answers (clarity, security, …)
- But a big one is performance: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs (n) because probably any algorithm is “plenty good” for small inputs (if n is 10, probably anything is fast enough)

Answer will be independent of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”
- Can do analysis before coding!
Analyzing code ("worst case")

Basic operations take "some amount of" constant time
- Arithmetic (fixed-width)
- Assignment
- Access one Java field or array index
- Etc.
(This is an approximation of reality: a very useful "lie".)

<table>
<thead>
<tr>
<th>Consecutive statements</th>
<th>Sum of time of each statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conditionals</td>
<td>Time of condition plus time of slower branch</td>
</tr>
<tr>
<td>Loops</td>
<td>Num iterations * time for loop body</td>
</tr>
<tr>
<td>Function Calls</td>
<td>Time of function’s body</td>
</tr>
<tr>
<td>Recursion</td>
<td>Solve recurrence equation</td>
</tr>
</tbody>
</table>
Example

// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    ???
}

Find an integer in a sorted array

2 3 5 16 37 50 73 75 126
Linear search

Find an integer in a sorted array

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    for (int i = 0; i < arr.length; ++i)
        if (arr[i] == k)
            return true;
    return false;
}
```

Best case:  
Worst case:
Linear search

Find an integer in a *sorted* array

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k)
{
    for (int i=0; i < arr.length; ++i)
        if (arr[i] == k)
            return true;
    return false;
}
```

Best case: 6 “ish” steps = $O(1)$
Worst case: 5 “ish” * (arr.length)  
           = $O(arr.length)$
Analyzing Recursive Code

• Computing run-times gets interesting with recursion
• Say we want to perform some computation recursively on a list of size n
  – Conceptually, in each recursive call we:
    • Perform some amount of work, call it \(w(n)\)
    • Call the function recursively with a smaller portion of the list

• So, if we do \(w(n)\) work per step, and reduce the problem size in the next recursive call by 1, we do total work:
  \[
  T(n) = w(n) + T(n-1)
  \]
• With some base case, like \(T(1)=5=O(1)\)
Example Recursive code: sum array

Recursive:
- Recurrence is some constant amount of work $O(1)$ done $n$ times

Recurrence Relation: $T(n) = O(1) + T(n-1)$

```java
int sum(int[] arr){
    return help(arr,0);
}
int help(int[]arr,int i) {
    if(i==arr.length)
        return 0;
    return arr[i] + help(arr,i+1);
}
```
Solving Recurrence Relations

• Say we have the following recurrence relation:
  \[ T(n) = 6 \text{ “ish”} + T(n-1) \]
  \[ T(1) = 9 \text{ “ish”} \] ← base case

• Now we just need to solve it; that is, reduce it to a closed form.

• Start by writing it out:
  \[ T(n) = 6 + T(n-1) \]
  \[ = 6 + 6 + T(n-2) \]
  \[ = 6 + 6 + 6 + T(n-3) \]
  \[ = 6k + T(n-k) \]
  \[ = 6 + 6 + 6 + \ldots + 6 + T(1) = 6 + 6 + 6 + \ldots + 6 + 9 \]
  \[ = 6k + 9, \text{ where } k \text{ is the } \# \text{ of times we expanded } T() \]

• We expanded it out \( n-1 \) times, so
  \[ T(n) = 6k + T(n-k) \]
  \[ = 6(n-1) + T(1) = 6(n-1) + 9 \]
  \[ = 6n + 8 = O(n) \]

Or When does \( n-k=1? \)
Answer: when \( k = n-1 \)
Binary search

Find an integer in a sorted array

– Can also be done non-recursively but “doesn’t matter” here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    return help(arr,k,0,arr.length);
}

boolean help(int[] arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;  // i.e., lo+(hi-lo)/2
    if(lo==hi) return false;
    if(arr[mid]==k) return true;
    if(arr[mid]< k) return help(arr,k,mid+1,hi);
    else          return help(arr,k,lo,mid);
}
```
Binary search

Best case: 9 “ish” steps = $O(1)$
Worst case: $T(n) = 10 \text{ “ish”} + T(n/2)$ where $n$ is $hi-lo$
- $O(\log n)$ where $n$ is $array.length$
- Solve recurrence equation to know that…

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    return help(arr, k, 0, arr.length);
}

boolean help(int[] arr, int k, int lo, int hi) {
    int mid = (hi+lo)/2;
    if (lo==hi) return false;
    if (arr[mid]==k) return true;
    if (arr[mid]< k) return help(arr, k, mid+1, hi);
    else return help(arr, k, lo, mid);
}
```
Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   \[ T(n) = 10 + T(n/2) \quad T(1) = 15 \]

2. “Expand” the original relation to find an equivalent general expression in terms of the number of expansions.

3. Find a closed-form expression by setting the number of expansions to a value which reduces the problem to a base case.
Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - \( T(n) = 10 + T(n/2) \quad T(1) = 15 \)

2. “Expand” the original relation to find an equivalent general expression in terms of the number of expansions.
   - \( T(n) = 10 + 10 + T(n/4) \)
     = \( 10 + 10 + 10 + T(n/8) \)
     = \( \ldots \)
     = \( 10k + T(n/(2^k)) \) (where \( k \) is the number of expansions)

3. Find a closed-form expression by setting the number of expansions to a value which reduces the problem to a base case
   - \( n/(2^k) = 1 \) means \( n = 2^k \) means \( k = \log_2 n \)
   - So \( T(n) = 10 \log_2 n + 15 \) (get to base case and do it)
   - So \( T(n) \) is \( O(\log n) \)
**Ignoring constant factors**

- So binary search is $O(\log n)$ and linear is $O(n)$
  - But which is faster?
  - Depending on constant factors and size of $n$, in a particular case, linear search could be faster….

- Could depend on constant factors
  - How *many* assignments, additions, etc. for each $n$
  - And could depend on size of $n$

- **But** there exists some $n_0$ such that for all $n > n_0$ binary search wins

- Let’s play with a couple plots to get some intuition…
Example

- Let’s try to “help” linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search
- Note: 600x still helpful for problems without logarithmic algorithms!
Another example: sum array

Two “obviously” linear algorithms: \( T(n) = O(1) + T(n-1) \)

Iterative:

```java
int sum(int[] arr){
    int ans = 0;
    for(int i=0; i<arr.length; ++i)
        ans += arr[i];
    return ans;
}
```

Recursive:

- Recurrence is \( c + c + \ldots + c \) for \( n \) times

```java
int sum(int[] arr){
    return help(arr,0);
}

int help(int[] arr, int i) {
    if(i==arr.length)
        return 0;
    return arr[i] + help(arr,i+1);
}
```
What about a **binary** version of sum?

```java
int sum(int[] arr){
    return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi){
    if(lo==hi) return 0;
    if(lo==hi-1) return arr[lo];
    int mid = (hi+lo)/2;
    return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is \( T(n) = O(1) + 2T(n/2) \)
- \( 1 + 2 + 4 + 8 + \ldots \) for \( \log n \) times
- \( 2^{(\log n)} - 1 \) which is proportional to \( n \) (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

“Obvious”: You can’t do better than \( O(n) \) – have to read whole array
Parallelism teaser

• But suppose we could do two recursive calls at the same time
  – Like having a friend do half the work for you!

```java
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if (lo == hi) return 0;
    if (lo == hi - 1) return arr[lo];
    int mid = (hi + lo) / 2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

• If you have as many “friends of friends” as needed, the recurrence is now
  \( T(n) = O(1) + \log n \cdot T(n/2) \)
  – \( O(\log n) \) : same recurrence as for `find`
Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

\[
T(n) = O(1) + T(n-1) \text{ \hspace{1cm} linear}
\]
\[
T(n) = O(1) + 2T(n/2) \text{ \hspace{1cm} linear}
\]
\[
T(n) = O(1) + T(n/2) \text{ \hspace{1cm} logarithmic}
\]
\[
T(n) = O(1) + 2T(n-1) \text{ \hspace{1cm} exponential}
\]
\[
T(n) = O(n) + T(n-1) \text{ \hspace{1cm} quadratic}
\]
\[
T(n) = O(n) + T(n/2) \text{ \hspace{1cm} linear}
\]
\[
T(n) = O(n) + 2T(n/2) \text{ \hspace{1cm} } O(n \log n)
\]

Note big-Oh can also use more than one variable

• Example: can sum all elements of an \(n\)-by-\(m\) matrix in \(O(nm)\)
Asymptotic notation

About to show formal definition, which amounts to saying:
1. Eliminate low-order terms
2. Eliminate coefficients

Examples:
- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$
Examples

True or false?

1. $4+3n$ is $O(n)$
2. $n+2\log n$ is $O(\log n)$
3. $\log n+2$ is $O(1)$
4. $n^{50}$ is $O(1.1^n)$

Notes:

• Do NOT ignore constants that are not multipliers:
  – $n^3$ is $O(n^2)$ : FALSE
  – $3^n$ is $O(2^n)$ : FALSE
• When in doubt, refer to the definition)
Examples

True or false?

1. 4+3n is O(n)  \hspace{1cm} True
2. n+2logn is O(logn)  \hspace{1cm} False
3. logn+2 is O(1)  \hspace{1cm} False
4. n^{50} is O(1.1^n)  \hspace{1cm} True

Notes:
• Do NOT ignore constants that are not multipliers:
  – n^3 is O(n^2) : FALSE
  – 3^n is O(2^n) : FALSE
• When in doubt, refer to the definition)
Big-Oh relates functions

We use $O$ on a function $f(n)$ (for example $n^2$) to mean the set of functions with asymptotic behavior less than or equal to $f(n)$

So $(3n^2+17)$ is in $O(n^2)$
- $3n^2+17$ and $n^2$ have the same asymptotic behavior

Confusingly, we also say/write:
- $(3n^2+17)$ is $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$
Formally Big-Oh

Definition: \( g(n) \) is in \( O(f(n)) \) iff there exist positive constants \( c \) and \( n_0 \) such that

\[
g(n) \leq c f(n) \quad \text{for all } n \geq n_0
\]

To show \( g(n) \) is in \( O(f(n)) \), pick a \( c \) large enough to “cover the constant factors” and \( n_0 \) large enough to “cover the lower-order terms”

- Example: Let \( g(n) = 3n^2 + 17 \) and \( f(n) = n^2 \)
  
  \( c = 5 \) and \( n_0 = 10 \) is more than good enough

This is “less than or equal to”

- So \( 3n^2 + 17 \) is also \( O(n^5) \) and \( O(2^n) \) etc.
Using the definition of Big-Oh (Example 1)

For $g(n) = 4n$ & $f(n) = n^2$, prove $g(n)$ is in $O(f(n))$

– A valid proof is to find valid $c$ & $n_0$
– When $n=4$, $g(n) = 16$ & $f(n) = 16$; this is the crossing over point
– So we can choose $n_0 = 4$, and $c = 1$

– Note: There are many possible choices:
  ex: $n_0 = 78$, and $c = 42$ works fine

The Definition: $g(n)$ is in $O(f(n))$ iff there exist positive constants $c$ and $n_0$ such that

$$g(n) \leq c \ f(n) \text{ for all } n \geq n_0.$$
Using the definition of Big-Oh (Example 2)

For $g(n) = n^4$ & $f(n) = 2^n$, prove $g(n)$ is in $O(f(n))$

- A valid proof is to find valid $c$ & $n_0$
- One possible answer: $n_0 = 20$, and $c = 1$

The Definition: $g(n)$ is in $O(f(n))$ iff there exist positive constants $c$ and $n_0$ such that

$$g(n) \leq c \cdot f(n) \text{ for all } n \geq n_0.$$
What’s with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called c)
- Consider:
  \[ g(n) = 7n + 5 \]
  \[ f(n) = n \]
- These have the same asymptotic behavior (linear), so \( g(n) \) is in \( O(f(n)) \) even though \( g(n) \) is always larger
- There is no positive \( n_0 \) such that \( g(n) \leq f(n) \) for all \( n \geq n_0 \)
- The ‘c’ in the definition allows for that:
  \[ g(n) \leq c f(n) \quad \text{for all} \quad n \geq n_0 \]
- To prove \( g(n) \) is in \( O(f(n)) \), have \( c = 12, n_0 = 1 \)
What you can drop

• Eliminate coefficients because we don’t have units anyway
  – $3n^2$ versus $5n^2$ doesn’t mean anything when we have not specified the cost of constant-time operations (can re-scale)

• Eliminate low-order terms because they have vanishingly small impact as $n$ grows

• Do NOT ignore constants that are not multipliers
  – $n^3$ is not $O(n^2)$
  – $3^n$ is not $O(2^n)$

(This all follows from the formal definition)
# Big Oh: Common Categories

*From fastest to slowest*

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(1)$</td>
<td>constant (same as $O(k)$ for constant $k$)</td>
</tr>
<tr>
<td>$O(\log n)$</td>
<td>logarithmic</td>
</tr>
<tr>
<td>$O(n)$</td>
<td>linear</td>
</tr>
<tr>
<td>$O(n \log n)$</td>
<td>“$n \log n$”</td>
</tr>
<tr>
<td>$O(n^2)$</td>
<td>quadratic</td>
</tr>
<tr>
<td>$O(n^3)$</td>
<td>cubic</td>
</tr>
<tr>
<td>$O(n^k)$</td>
<td>polynomial (where $k$ is any constant $&gt; 1$)</td>
</tr>
<tr>
<td>$O(k^n)$</td>
<td>exponential (where $k$ is any constant $&gt; 1$)</td>
</tr>
</tbody>
</table>

Usage note: “exponential” does not mean “grows really fast”, it means “grows at rate proportional to $k^n$ for some $k>1$”
More Asymptotic Notation

- **Upper bound**: $O(f(n))$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \leq c f(n)$ for all $n \geq n_0$

- **Lower bound**: $\Omega(f(n))$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(f(n))$ if there exist constants $c$ and $n_0$ such that $g(n) \geq c f(n)$ for all $n \geq n_0$

- **Tight bound**: $\Theta(f(n))$ is the set of all functions asymptotically equal to $f(n)$
  - Intersection of $O(f(n))$ and $\Omega(f(n))$ (use different $c$ values)
Regarding use of terms

A common error is to say \( O(f(n)) \) when you mean \( \theta(f(n)) \)

- People often say O() to mean a tight bound
- Say we have \( f(n) = n \); we could say \( f(n) \) is in \( O(n) \), which is true, but only conveys the upper-bound
- Since \( f(n) = n \) is also \( O(n^5) \), it’s tempting to say “this algorithm is exactly \( O(n) \)”
- Somewhat incomplete; instead say it is \( \theta(n) \)
- That means that it is not, for example \( O(\log n) \)

Less common notation:

- “little-oh”: like “big-Oh” but strictly less than
  - Example: sum is \( o(n^2) \) but not \( o(n) \)
- “little-omega”: like “big-Omega” but strictly greater than
  - Example: sum is \( \omega(\log n) \) but not \( \omega(n) \)
What we are analyzing

• The most common thing to do is give an $O$ or $\theta$ bound to the worst-case running time of an algorithm

• Example: True statements about binary-search algorithm
  – Common: $\theta(\log n)$ running-time in the worst-case
  – Less common: $\theta(1)$ in the best-case (item is in the middle)
  – Less common: Algorithm is $\Omega(\log \log n)$ in the worst-case (it is not really, really, really fast asymptotically)
  – Less common (but very good to know): the find-in-sorted-array problem is $\Omega(\log n)$ in the worst-case
    • No algorithm can do better (without parallelism)
    • A problem cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean there exists an algorithm
Other things to analyze

• Space instead of time
  – Remember we can often use space to gain time

• Average case
  – Sometimes only if you assume something about the distribution of inputs
    • See CSE312 and STAT391
  – Sometimes uses randomization in the algorithm
    • Will see an example with sorting; also see CSE312
  – Sometimes an amortized guarantee
    • Will discuss in a later lecture
Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)
Big-Oh Caveats

• Asymptotic complexity (Big-Oh) focuses on behavior for large \( n \) and is independent of any computer / coding trick
  – But you can “abuse” it to be misled about trade-offs
  – Example: \( n^{1/10} \) vs. \( \log n \)
    • Asymptotically \( n^{1/10} \) grows more quickly
    • But the “cross-over” point is around \( 5 \times 10^{17} \)
    • So if you have input size less than \( 2^{58} \), prefer \( n^{1/10} \)
• Comparing \( O() \) for small \( n \) values can be misleading
  – Quicksort: \( O(n\log n) \) (expected)
  – Insertion Sort: \( O(n^2) \) (expected)
  – Yet in reality Insertion Sort is faster for small \( n \)’s
  – We’ll learn about these sorts later
Addendum: Timing vs. Big-Oh?

• At the core of CS is a backbone of theory & mathematics
  – Examine the algorithm itself, mathematically, not the implementation
  – Reason about performance as a function of n
  – Be able to mathematically prove things about performance
• Yet, timing has its place
  – In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  – Ex: Benchmarking graphics cards
  – We will do some timing in project 3 (and in 2, a bit)
• Evaluating an algorithm? Use asymptotic analysis
• Evaluating an implementation of hardware/software? Timing can be useful