CSE332: Data Abstractions

Lecture 2: Math Review; Algorithm Analysis

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Announcements

- Project 1 – phase A due next Wed
- Homework 1 - due next Friday

Today

- Finish discussing queues
- Review math essential to algorithm analysis
  - Proof by induction
  - Bit patterns
  - Powers of 2
  - Exponents and logarithms
- Begin analyzing algorithms
  - Using asymptotic analysis (continue next time)

Mathematical induction

Suppose *P(n)* is some predicate (involving integer *n*)

- Example: $n \geq n/2 + 1$ (for all n ≥ 2)

To prove *P(n)* for all integers $n \geq c$, it suffices to prove

1. *P(c)* – called the "basis" or "base case"
2. If *P(k)* then *P(k+1)* – called the "induction step" or "inductive case"

We will use induction:

To show an algorithm is correct or has a certain running time *no matter how big a data structure or input value is*

(Our "*n*" will be the data structure or input size.)

***P(n)* = "the sum of the first *n* powers of 2 (starting at 2^0) is 2^n-1"**

Inductive Proof Example

Theorem: *P(n)* holds for all $n \geq 1$

Proof: By induction on *n*

- Base case, n=1: Sum of first power of 2 is 2^0, which equals 1.
  And for n=1, 2^n-1 equals 1.
- Inductive case:
  - Inductive hypothesis: Assume the sum of the first *k* powers of 2 is 2^k-1
  - Show, given the hypothesis, that the sum of the first (*k*+1) powers of 2 is $2^{k+1}-1$

From our inductive hypothesis we know:

$$1+2+4+...+2^{k-1}=2^k-1$$

Add the next power of 2 to both sides…

$$1+2+4+...+2^{k-1}+2^k=2^k-1+2^k$$

We have what we want on the left; massage the right a bit

$$1+2+4+...+2^{k-1}+2^k=2(2^k)-1=2^{k+1}-1$$

Note for homework

Proofs by induction will come up a fair amount on the homework

When doing them, be sure to state each part clearly:

- What you're trying to prove
- The base case
- The inductive case
- The inductive hypothesis
  - In many inductive proofs, you'll prove the inductive case by just starting with your inductive hypothesis, and playing with it a bit, as shown above

N bits can represent how many things?

# Bits	Patterns	# of patterns
1		
2		

Powers of 2

- A bit is 0 or 1
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 (“about a thousand”, kilo in CSE speak)
- 2^{20} is “about a million”, mega in CSE speak
- 2^{30} is “about a billion”, giga in CSE speak

Java: an **`int`** is 32 bits and signed, so “max int” is “about 2 billion”

a **`long`** is 64 bits and signed, so “max long” is 2^{63}-1

Therefore…

Could give a unique id to…

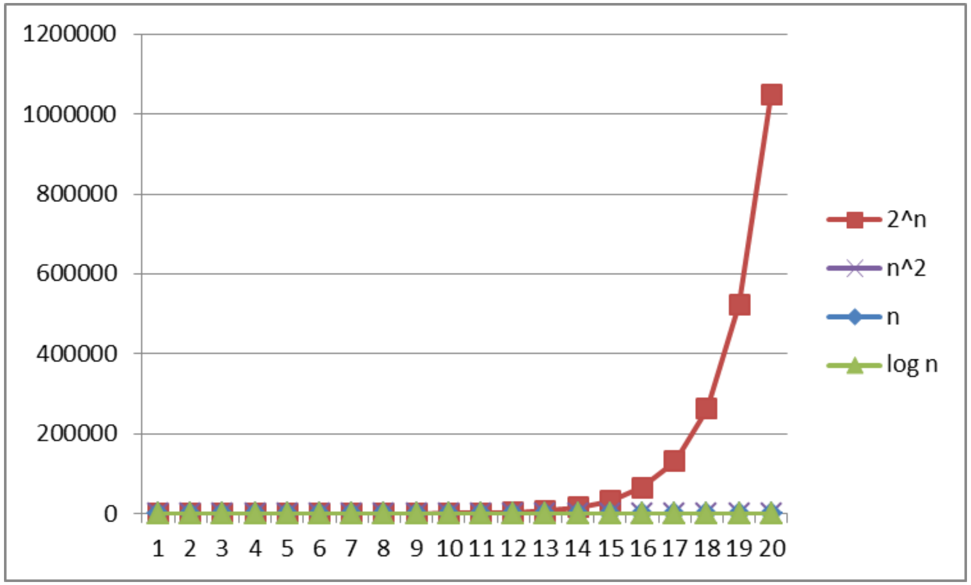
- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

So if a password is 128 bits long and randomly generated, do you think you could guess it?

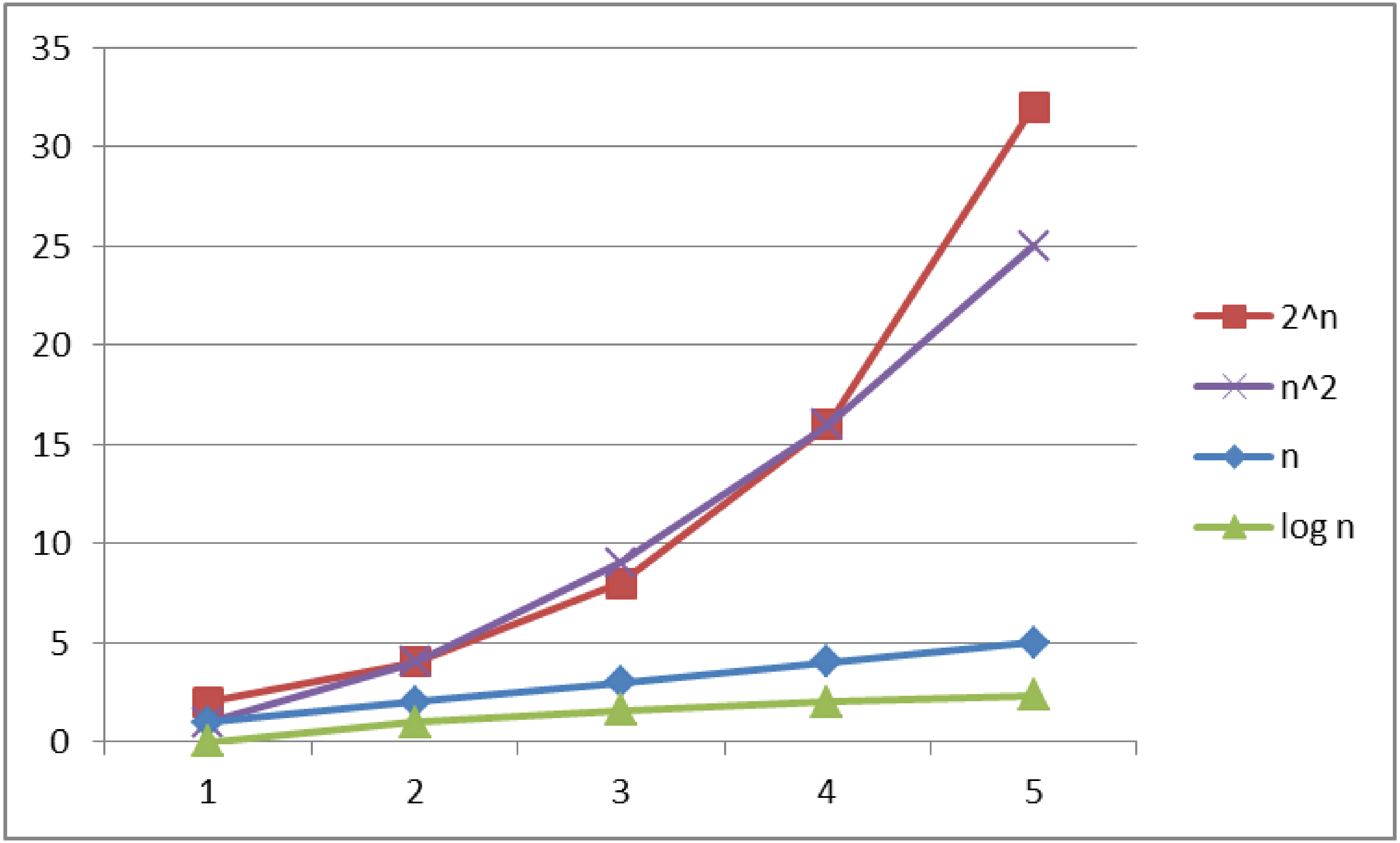
Logarithms and Exponents

- Since so much is binary in CS, **`log`** almost always means $\mathbf{log_2}$
- Definition: $\mathbf{log_2\ x = y}$ if $\mathbf{x = 2^y}$
- So, $\mathbf{log_2}$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

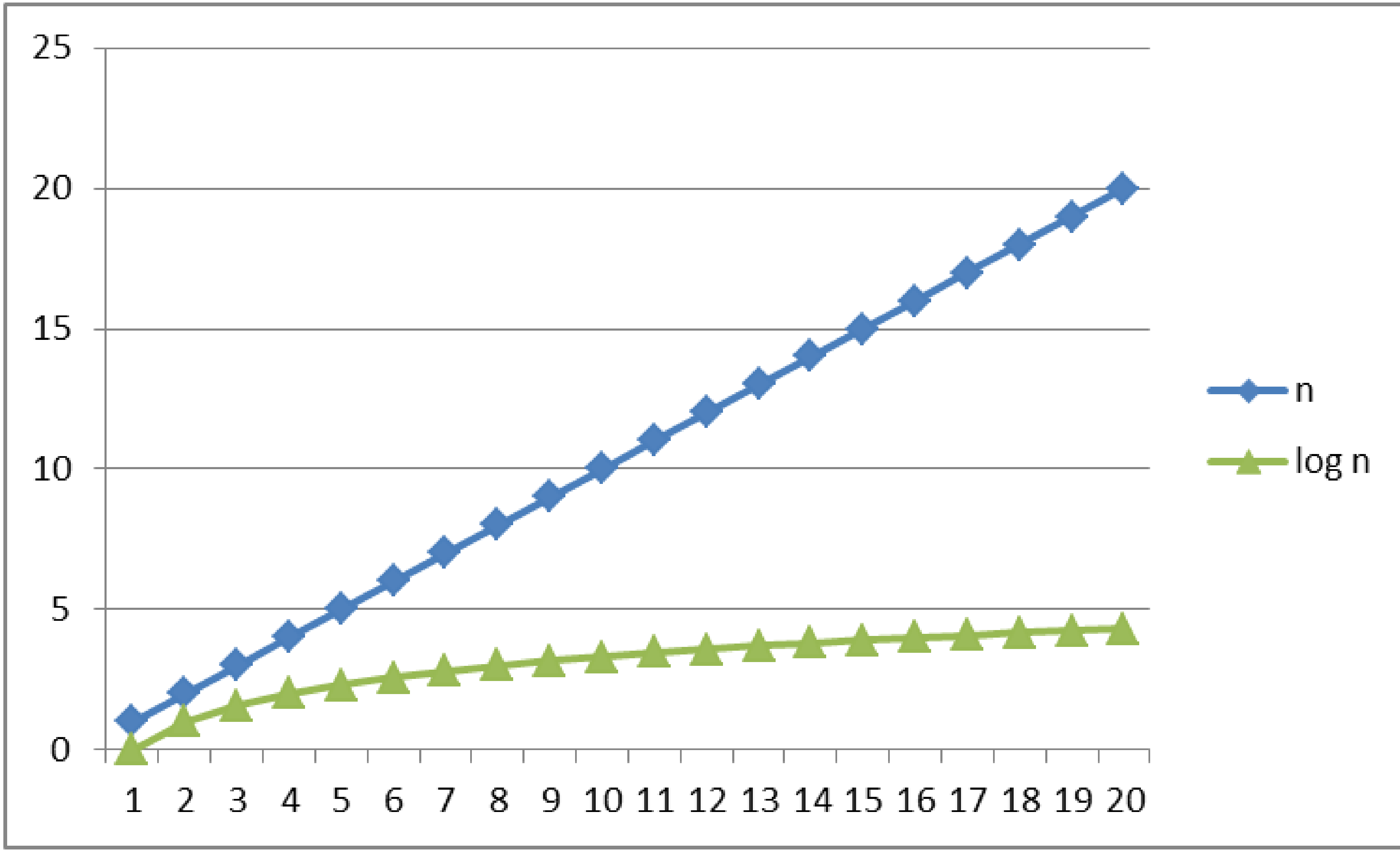
See Excel file for plot data – play with it!



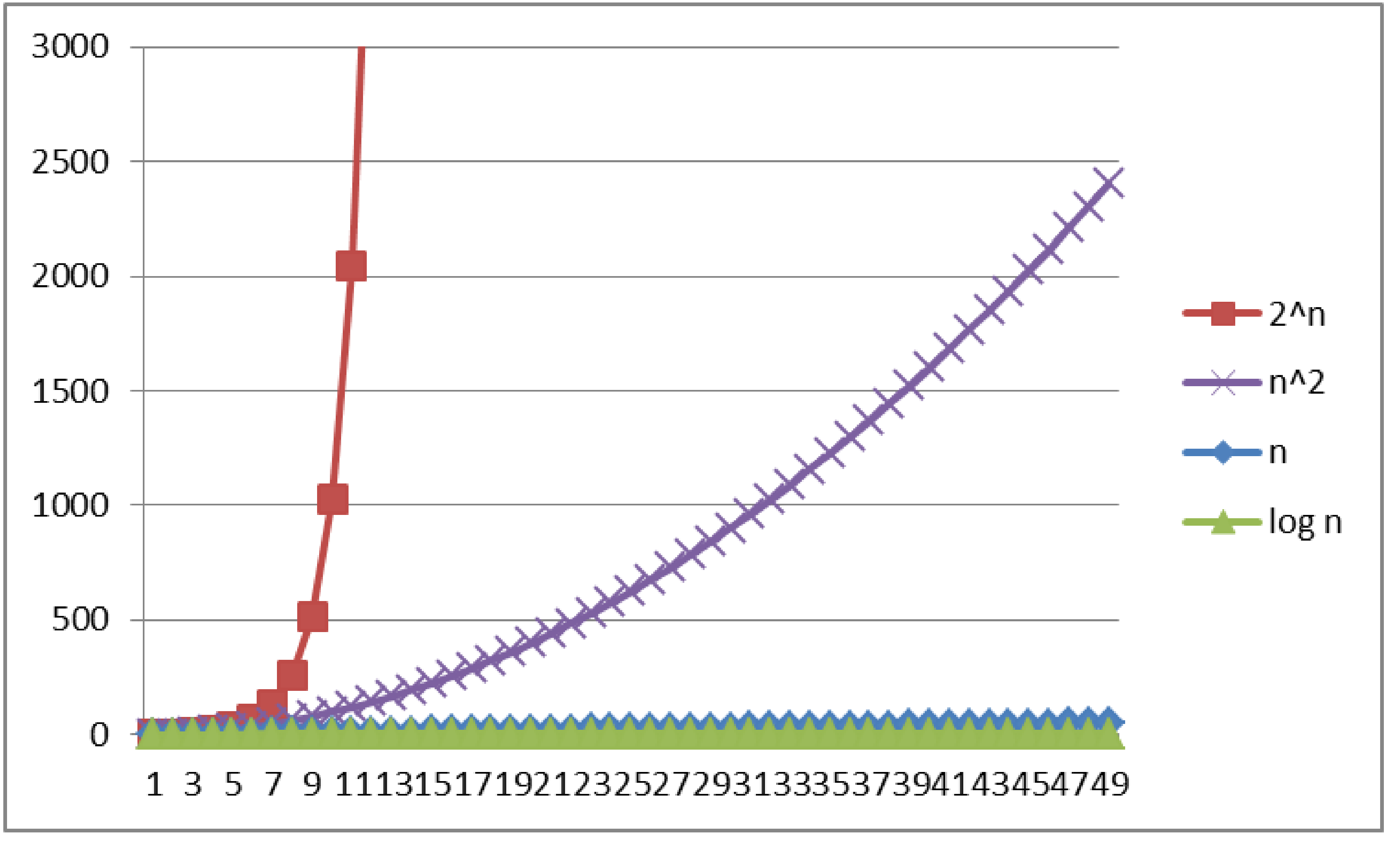
Logarithms and Exponents



Logarithms and Exponents



Logarithms and Exponents



Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So **`log(N`**^{k}**`)= k log N`**
- **`log(A/B) = log A – log B`**
- **`x =`** $\log_2 2^x$
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows fast
  - Ex: $$\log_2 \log_2 4billion \sim \log_2 \log_2 2^{32} = \log_2 32 = 5$$
- **`(log x)(log x)`** is written **`log`**^{2}**`x`**
  - It is greater than **`log x`** for all **`x > 2`**

Log base doesn't matter (much)

"Any base *B* log is equivalent to base 2 log within a constant factor"

- And we are about to stop worrying about constant factors!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general, we can convert log bases via a constant multiplier
- Say, to convert from base A to base B:

$$\log_B x = (\log_A x) / (\log_A B)$$

Algorithm Analysis

As the “size” of an algorithm’s input grows
(integer, length of array, size of queue, etc.):

- How much longer does the algorithm take (time)
- How much more memory does the algorithm need (space)

Because the curves we saw are so different, we often only care about “which curve we are like”

Separate issue: Algorithm *correctness* – does it produce the right answer for all inputs

- Usually more important, naturally

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns…

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns 3N(N+1)/2
- Proof: By induction on *n*
  - *P(n)* = after outer for-loop executes *n* times, **`x`** holds 3n(n+1)/2
  - Base: n=0, returns 0
  - Inductive: From *P(k)*, **`x`** holds 3k(k+1)/2 after *k* iterations. Next iteration adds 3(k+1), for total of 3k(k+1)/2 + 3(k+1) = (3k(k+1) + 6(k+1))/2 = (k+1)(3k+6)/2 = 3(k+1)(k+2)/2

Example

- How long does this pseudocode run?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Running time: For any N ≥ 0,
  - Assignments, additions, returns take “1 unit time”
  - Loops take the sum of the time for their iterations

- So: 2 + 2*(number of times inner loop runs)
  - And how many times is that?

Example

- How long does this pseudocode run?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- How many times does the inner loop run?

Example

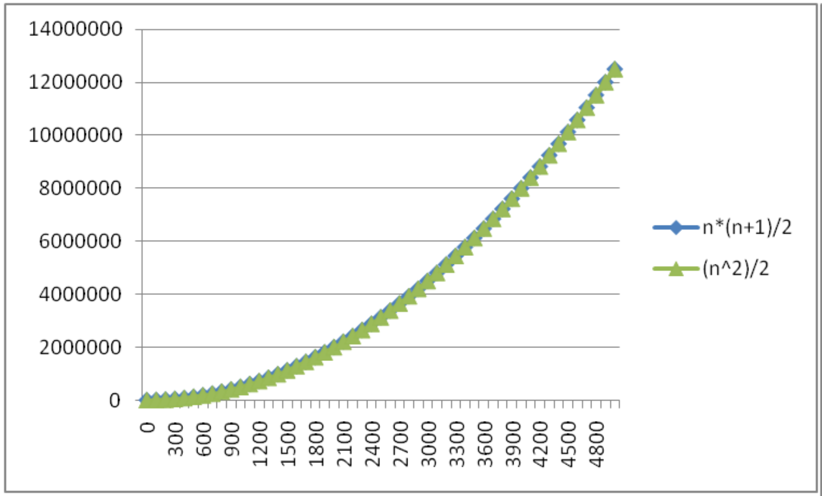
- How long does this pseudocode run?

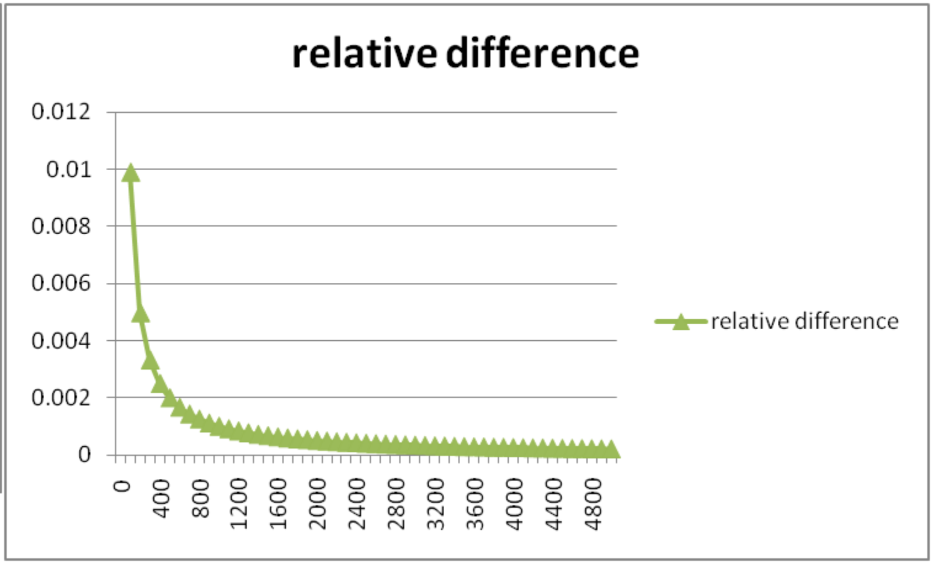
```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- The total number of loop iterations is $N*(N+1)/2$
  - This is a very common loop structure, worth memorizing
  - This is *proportional to* N^2 , and we say $O(N^2)$, “big-Oh of”
    - For large enough N, the N and constant terms are irrelevant, as are the first assignment and return
    - See plot… $N*(N+1)/2$ vs. just $N^2/2$

Lower-order terms don't matter

N*(N+1)/2 vs. just $N^2/2$

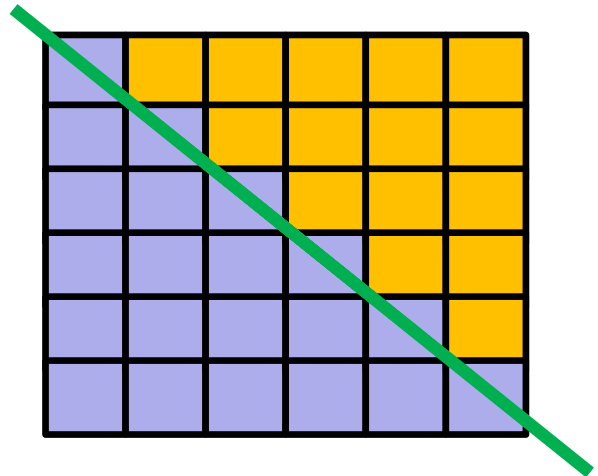




Geometric interpretation

$$\sum_{i=1}^{N} i = N*N/2+N/2$$

```
for i=1 to N do
  for j=1 to i do
     // small work
```



- Area of square: N*N
- Area of lower triangle of square: N*N/2
- Extra area from squares crossing the diagonal: N*1/2
- As N grows, fraction of “extra area” compared to lower triangle goes to zero (becomes insignificant)

Recurrence Equations

- For running time, what the loops did was irrelevant, it was how many times they executed.
- Running time as a function of input size n (here loop bound):

  $$T(n) = n + T(n-1)$$

  (and $T(0) = 2$ish, but usually implicit that $T(0)$ is some constant)
- Any algorithm with running time described by this formula is $O(n^2)$
- "Big-Oh" notation also ignores the constant factor on the high-order term, so $3N^2$ and $17N^2$ and $(1/1000)\ N^2$ are all $O(N^2)$
  - As N grows large enough, no smaller term matters
  - Next time: Many more examples + formal definitions

Big-O: Common Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic
$O(n)$	linear
$\mathrm{O}(\mathrm{n}\ \texttt{log}\ n)$	"n $\texttt{log}$ n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant > 1)
$O(k^n)$	exponential (where k is any constant > 1)

"exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"