CSE332: Data Abstractions

Lecture 18: Parallel Sort

James Fogarty

Winter 2012

Including slides developed in part by
Ruth Anderson, James Fogarty, Dan Grossman

Reductions

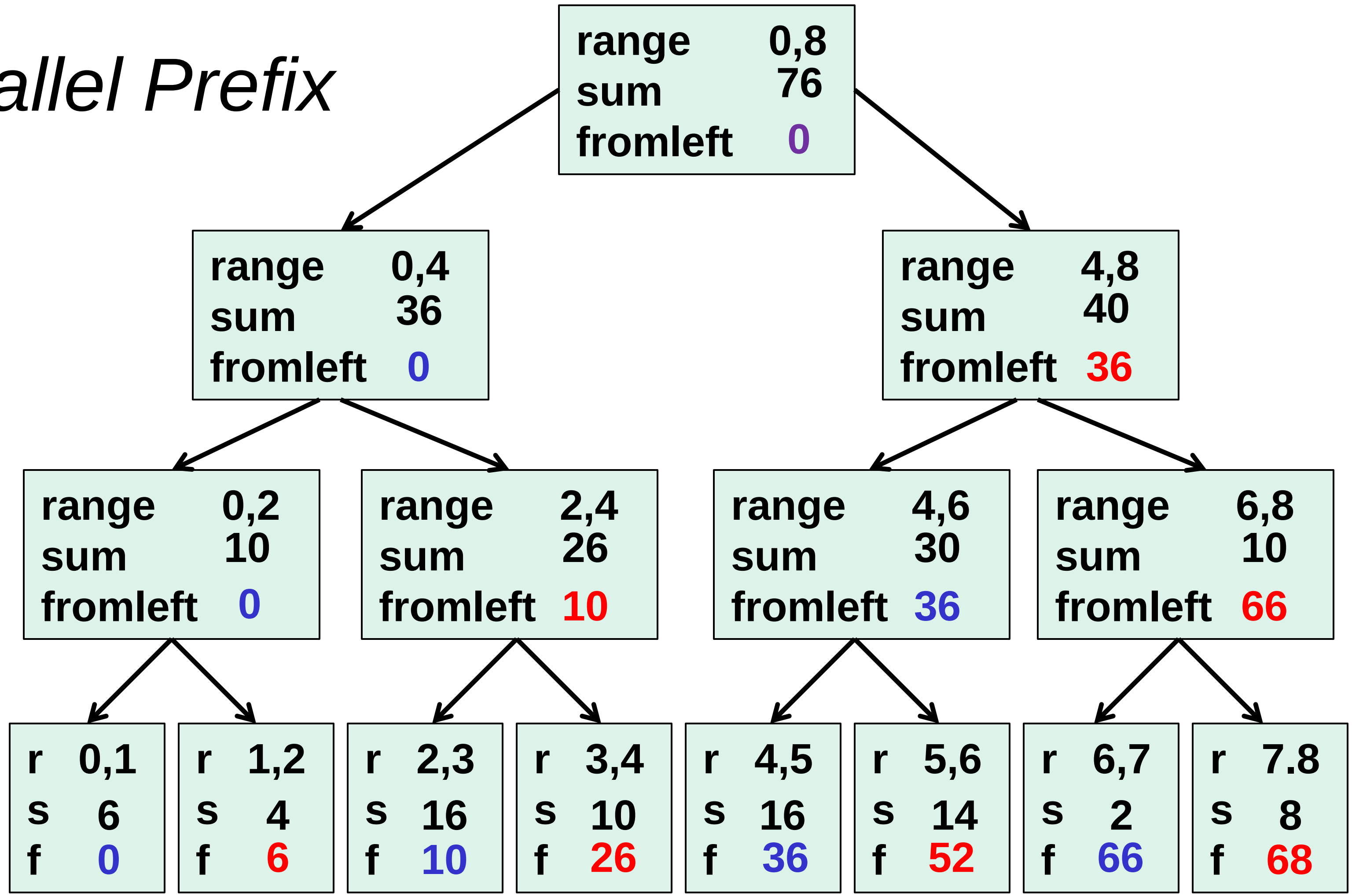
- Computations of this form are called reductions
- Produce single answer from collection via an associative operator
  - Examples: max, count, leftmost, rightmost, sum, …
  - Non-example: median
- Recursive results don’t have to be single numbers or strings. They can be arrays or objects with multiple fields.
  - Example: Histogram of test results is a variant of sum
- But some things are inherently sequential
  - How we process **`arr[i]`** may depend entirely on the result of processing **`arr[i-1]`**

Maps and Data Parallelism

- A map operates on each element of a collection independently to create a new collection of the same size
  - No combining results
  - For arrays, this is so trivial some hardware has direct support
- Canonical example: Vector addition

```
int[] vector_add(int[] arr1, int[] arr2){
  assert (arr1.length == arr2.length);
  result = new int[arr1.length];
  FORALL(i=0; i < arr1.length; i++) {
    result[i] = arr1[i] + arr2[i];
  }
  return result;
}
```

Parallel Prefix



`input`	6	4	16	10	16	14	2	8
`output`	6	10	26	36	52	66	68	76

Pack

[Non-standard terminology, **`filter`** does not emphasize stability]

Given an array **`input`**,
produce an array **`output`**
containing only elements such that **`f(elt)`** is **`true`**

```
Example: input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
         f: is elt > 10
         output [17, 11, 13, 19, 24]
```

Parallelizable

- Finding elements for the output is easy
- But getting them in the right place seems hard

Pack as Map, Parallel Prefix, Map

1. Parallel map to compute a bit-vector for true elements

```
input   [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits    [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```

Quicksort Review

Recall quicksort was sequential, in-place, expected time $O(n \log n)$

		Best / expected case *work*
1.	**Pick a pivot element**	**O(1)**
2.	**Partition all the data into:**	**O(n)**
	A. The elements less than the pivot	
	B. The pivot	
	C. The elements greater than the pivot	
3.	**Recursively sort A and C**	**2T(n/2)**

How should we parallelize this?

Quicksort

		Best / expected case *work*
1.	**Pick a pivot element**	**O(1)**
2.	**Partition all the data into:**	**O(n)**
	A. The elements less than the pivot	
	B. The pivot	
	C. The elements greater than the pivot	
3.	**Recursively sort A and C**	**2T(n/2)**

Easy: Do the two recursive calls in parallel

- Work: unchanged $O(n \log n)$
- Span: $T(n) = O(n) + T(n/2) = O(n) + O(n/2) + T(n/4) = O(n)$
- So parallelism is $O(\log n)$ (i.e., work / span)

Doing Better

- $O(\mathtt{log}\ n)$ speed-up with infinite number of processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30 times faster
- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong
  - But we need auxiliary storage (will no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law and the long-term situation
- Already have everything we need to parallelize the partition

Parallel Partition with Auxiliary Storage

Partition all the data into:

A. The elements less than the pivot

B. The pivot

C. The elements greater than the pivot

- This is just two packs
  - We know a pack is $O(n)$ work, $O(\texttt{log}\ n)$ span
  - Pack elements less than pivot into left side of **`aux`** array
  - Pack elements greater than pivot into right size of **`aux`** array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once
    - But no effect on asymptotic complexity

Analysis

With $O(\mathtt{log}\ n)$ span for partition, the total span for quicksort is

$$
\begin{aligned}
T(n) &= O(\mathtt{log}\ n) + T(n/2) \\
&= O(\mathtt{log}\ n) + O(\mathtt{log}\ n/2) + T(n/4) \\
&= O(\mathtt{log}\ n) + O(\mathtt{log}\ n/2) + O(\mathtt{log}\ n/4) + T(n/8) \\
&\dots \\
&= O(\mathtt{log}^2\ n)
\end{aligned}
$$

So parallelism (work / span) is $O(n / \mathtt{log}\ n)$

Example

- Step 1: pick pivot as median of three

8	1	4	9	0	3	5	2	7	6

- Steps 2a and 2c (combinable):
  pack less than and pack greater than into a second array
  - Fancy parallel prefix to pull this off not shown

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel
  - Can sort back into original array (swapping like in mergesort)

Mergesort

Recall mergesort: sequential, not-in-place, worst-case $O(n\ \mathtt{log}\ n)$

1. **Sort left half and right half** **2T(n/2)**
2. **Merge results** **O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the span to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, parallelism is $O(\mathtt{log}\ n)$
- To do better, need to parallelize the merge
  - The trick this time will not use parallel prefix

Parallelizing the Merge

Need to merge two *sorted* subarrays (may not have the same size)

0	1	4	8	9

2	3	5	6	7

Idea: Suppose the larger subarray has *n* elements. In parallel:

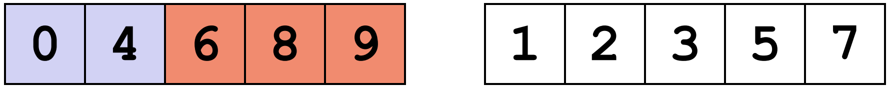
- merge the first *n*/2 elements of the larger half
  with the “appropriate” elements of the smaller half
- merge the second *n*/2 elements of the larger half
  with the remainder of the smaller half

Parallelizing the Merge

0	4	6	8	9

1	2	3	5	7

Parallelizing the Merge



1. Get median of bigger half:

Parallelizing the Merge

0	4	6	8	9

1	2	3	5	7

1. Get median of bigger half: $O(1)$ to compute middle index

Parallelizing the Merge

0	4	6	8	9

1	2	3	5	7

1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value:

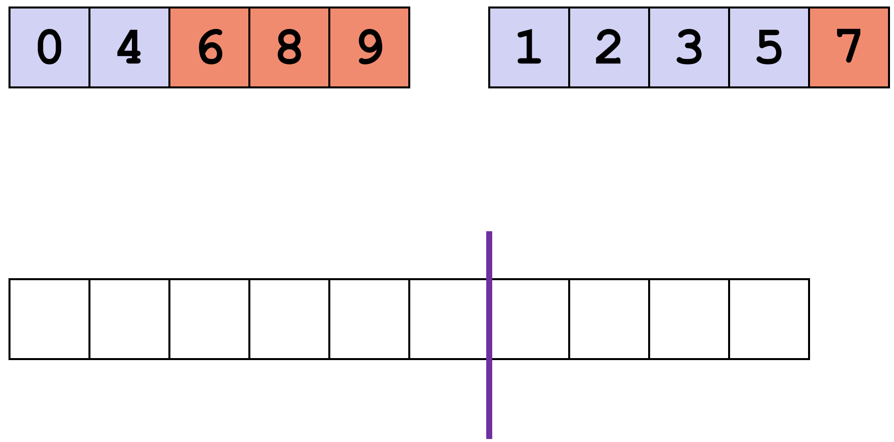
Parallelizing the Merge

0	**4**	**6**	**8**	**9**

1	**2**	**3**	**5**	**7**

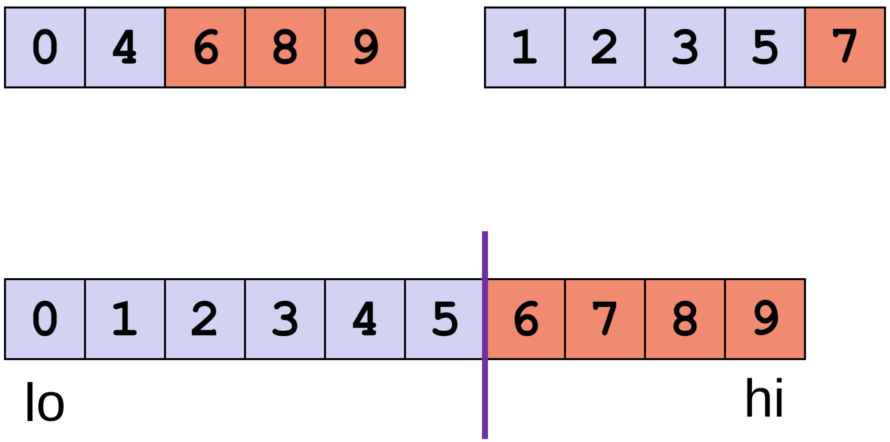
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value:
   $O(\texttt{log}\ n)$ to do binary search on the sorted small half

Parallelizing the Merge



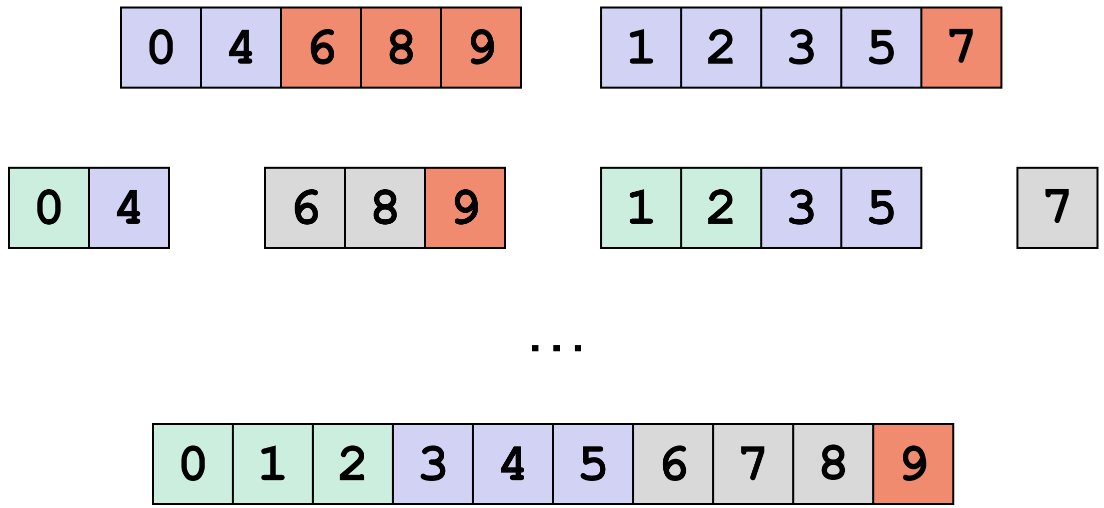
1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value:
   O(**`log`** *n*) to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: *O*(1)

Parallelizing the Merge



1. Get median of bigger half: *O*(1) to compute middle index
2. Find how to split the smaller half at the same value: *O*(`log` *n*) to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: *O*(1)
4. Do two submerges in parallel

The Recursion



When we do each merge in parallel,
we split the bigger array in half,
and use binary search to split the smaller array,
in base case we do the copy

Analysis

- Sequential recurrence for mergesort:

  $T(n) = 2T(n/2) + O(n)$ which is $O(n \log n)$

- Doing the two recursive calls in parallel but a sequential merge:

  work: same as sequential span: $T(n)=1T(n/2)+O(n)$ which is $O(n)$

- Parallel merge makes work and span harder to compute
  – Each merge step does an extra $O(\log n)$ binary search to find how to split the smaller subarray
  – To merge n elements total, do two smaller merges of possibly different sizes
  – But worst-case split is $(1/4)n$ and $(3/4)n$
    - When subarrays same size and “smaller” splits “all” / “none”

Analysis

For just a parallel merge of n elements:

- Span is $T(n) = T(3n/4) + O(\texttt{log}\ n)$, which is $O(\texttt{log}^2\ n)$
- Work is $T(n) = T(3n/4) + T(n/4) + O(\texttt{log}\ n)$ which is $O(n)$
- Neither bound is immediately obvious, but “trust us”

So for mergesort with parallel merge overall:

- Span is $T(n) = 1T(n/2) + O(\texttt{log}^2\ n)$, which is $O(\texttt{log}^3\ n)$
- Work is $T(n) = 2T(n/2) + O(n)$, which is $O(n\ \texttt{log}\ n)$

So parallelism (work / span) is $O(n / \texttt{log}^2\ n)$

- Not quite as good as quicksort’s $O(n / \texttt{log}\ n)$
  - But worst-case guarantee
- And as always this is just the asymptotic result

Toward Sharing Resources

Have been studying parallel algorithms using fork-join

- Lower span via parallel tasks

Algorithms all had a very simple *structure* to avoid race conditions

- Each thread had memory "only it accessed"
  - Example: array sub-range
- Or used **`fork`** and **`join`** as contract for who "had" memory
  - On **`fork`**, "loan" some memory to "forkee" and do not access that memory again until after **`join`** on the "forkee"

Strategy will not work well when:

- Memory accessed by threads is overlapping or unpredictable
- Threads are doing independent tasks needing access to same resources (as opposed to implementing the same algorithm)

Concurrent Programming

Concurrency:

Correctly and efficiently managing access to shared resources from multiple possibly-simultaneous clients

Requires *coordination*, particularly synchronization to avoid incorrect simultaneous access: make somebody *block*

- **`join`** is not what we want
- Want to block until another thread is “done with what we need”, not the more extreme “until completely done executing”

Even correct concurrent applications are usually highly non-deterministic: how threads are scheduled affects what each thread sees in its different operations

- non-repeatability complicates testing and debugging

Examples

Multiple threads:

1. Processing different bank-account operations
   – What if 2 threads change the same account at the same time?

2. Using a shared cache of recent files (e.g., hashtable)
   – What if 2 threads insert the same file at the same time?

3. Creating a pipeline with a queue for handing work to next thread in sequence (i.e., a virtual assembly line)?
   – What if enqueuer and dequeuer
     adjust a circular array queue at the same time?

Why Threads?

Unlike parallelism, not about implementing algorithms faster

But threads still useful for:

- *Code structure for responsiveness*
  - Respond to GUI events in one thread while another thread is performing an expensive computation
- *Processor utilization (mask I/O latency)*
  - If 1 thread “goes to disk,” have something else to do
- *Failure isolation*
  - Convenient structure if want to *interleave* multiple tasks and do not want an exception in one to stop the other