CSE332: Data Abstractions

Lecture 12: Graphs

James Fogarty

Winter 2012

Including slides developed in part by
Ruth Anderson, James Fogarty, Dan Grossman, Richard Ladner, Steve Seitz

Graphs

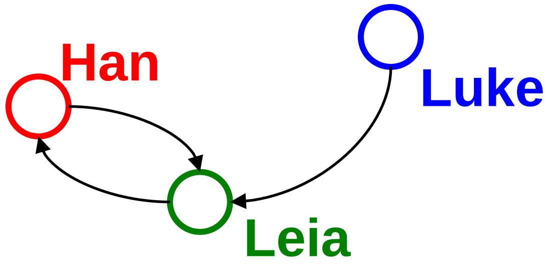
- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

- A graph is a pair

  `G = (V,E)`
  - A set of vertices, also known as nodes

    $V = \{v_1, v_2, \ldots, v_n\}$
  - A set of edges

    $E = \{e_1, e_2, \ldots, e_m\}$
    - Each edge e_i is a pair of vertices (v_j, v_k)
    - An edge "connects" the vertices




```
V = {Han,Leia,Luke}
E = {(Luke,Leia),
     (Han,Leia),
     (Leia,Han)}
```

- Graphs can be directed or undirected

An ADT?

- Can think of graphs as an ADT with operations like **isEdge((v_j,v_k))**
- But it is unclear what the "standard operations" are
- Instead we tend to develop algorithms over graphs and then use data structures that are efficient for those algorithms
- Many important problems can be solved by:
  1. Formulating them in terms of graphs
  2. Applying a standard graph algorithm

- To make the formulation easy and standard, we have a lot of *standard terminology* for graphs

Some Graphs

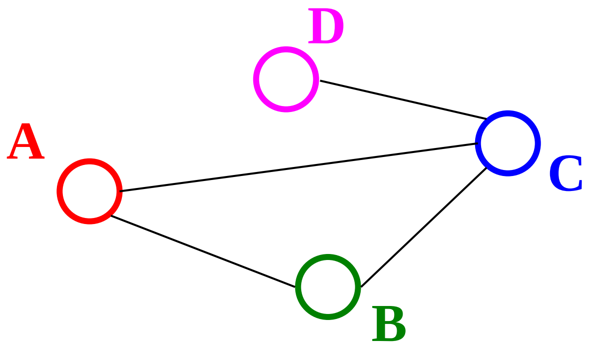
For each, what are the vertices and what are the edges?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
- Course pre-requisites
- …

Core algorithms that work across such domains is why we are CSE

Undirected Graphs

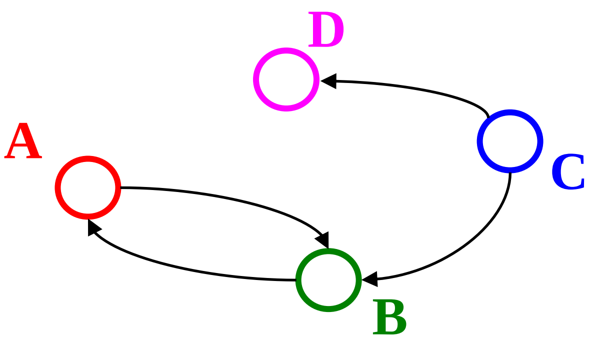
- In undirected graphs, edges have no specific direction
  - Edges are always "two-way"



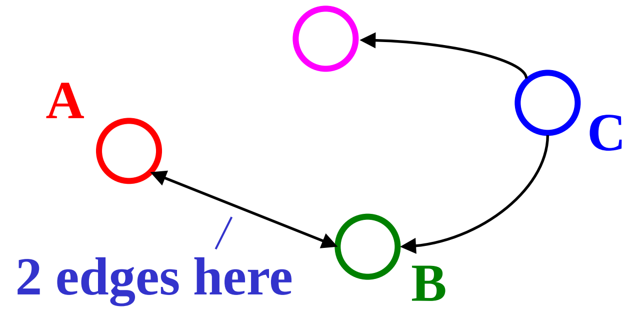
- Thus, **`(u,v)`** $\in$ **`E`** implies **`(v,u)`** $\in$ **`E`**.
  - Only one of these edges needs to be in the set
  - The other is implicit, so normalize how you check for it
- Degree of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices

Directed Graphs

- In directed graphs (a.k.a. digraphs), edges have a direction



or

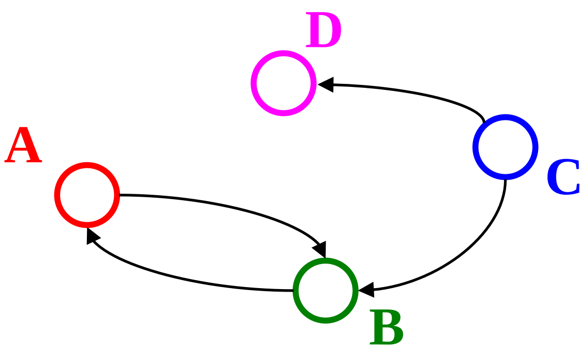


- Thus, $(u,v) \in E$ does *not* imply $(v,u) \in E$.
  - Let $(u,v) \in E$ mean u → v
  - Call **u** the source and **v** the destination
- In-Degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination
- Out-Degree of a vertex: number of out-bound edges, i.e., edges where the vertex is the source

Self-Edges, Connectedness

- A self-edge a.k.a. a loop edge is of the form **`(u,u)`**
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of zero
- A graph does not have to be connected
  - Even if every node has non-zero degree
  - More discussion of this to come

More Notation



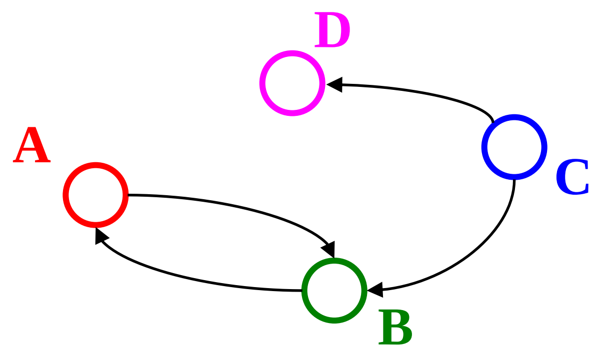
For a graph **`G = (V,E)`**:

- **`|V|`** is the number of vertices
- **`|E|`** is the number of edges
  - Minimum?
  - Maximum for undirected?
  - Maximum for directed?

`V = {A, B, C, D}`
`E = {(C, B),`
`(A, B),`
`(B, A)`
`(C, D)}`

- If **`(u,v)`** $\in$ **`E`**
  - Then **`v`** is a neighbor of **`u`** (i.e., **`v`** is adjacent to **`u`**)
  - Order matters for directed edges
    - **`u`** is not adjacent to **`v`** unless **`(v,u)`** $\in$ **`E`**

More Notation



For a graph $\mathbf{G = (V,E)}$:

- $|\mathbf{V}|$ is the number of vertices
- $|\mathbf{E}|$ is the number of edges
  - Minimum? 0
  - Maximum for undirected? $|V||V+1|/2 \in O(|V|^2)$
  - Maximum for directed? $|V|^2 \in O(|V|^2)$

    (assuming self-edges allowed, else subtract $|V|$)
- If $(\mathbf{u},\mathbf{v}) \in \mathbf{E}$
  - Then $\mathbf{v}$ is a neighbor of $\mathbf{u}$ (i.e., $\mathbf{v}$ is adjacent to $\mathbf{u}$)
  - Order matters for directed edges
    - $\mathbf{u}$ is not adjacent to $\mathbf{v}$ unless $(\mathbf{v},\mathbf{u}) \in \mathbf{E}$

Examples again

Which would use directed edges?

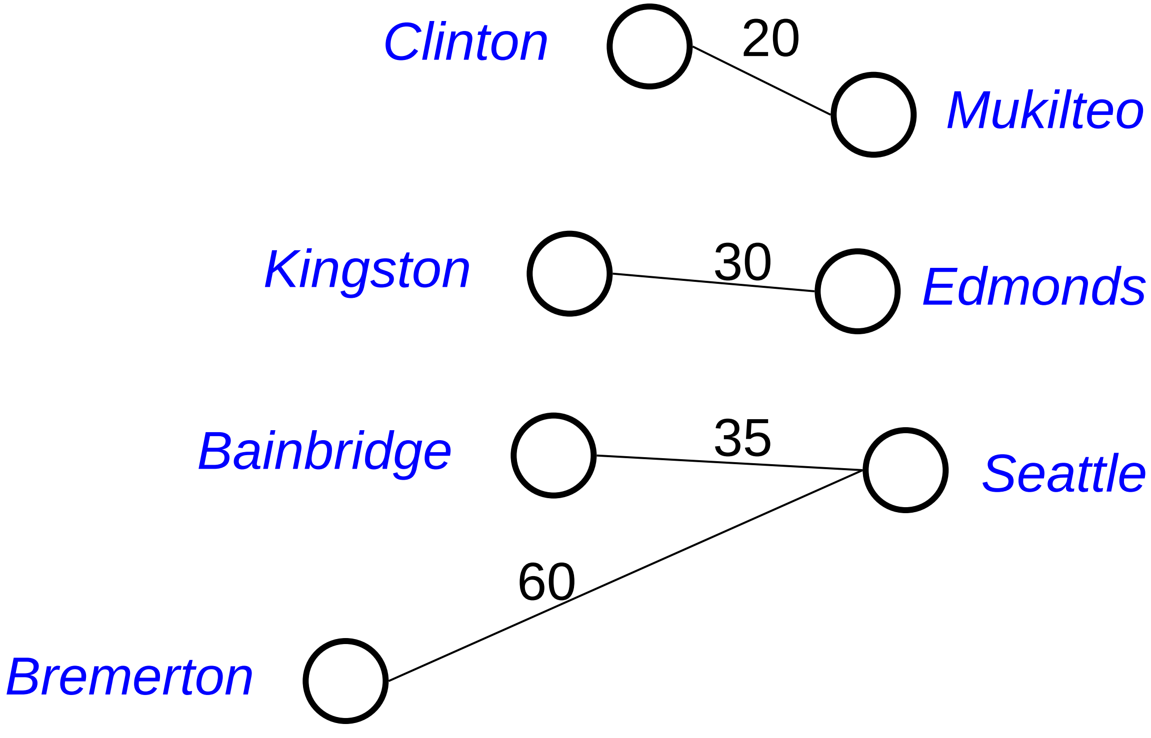
Which would have self-edges?

Which could have 0-degree nodes?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
- Course pre-requisites
- …

Weighted Graphs

- In a weighted graph, each edge has a weight a.k.a. cost
  - Typically numeric (our examples use ints, but not required)
  - *Orthogonal* to whether graph is directed
  - Some graphs allow *negative weights*; many do not



Examples

What, if anything, might weights represent for each of these?

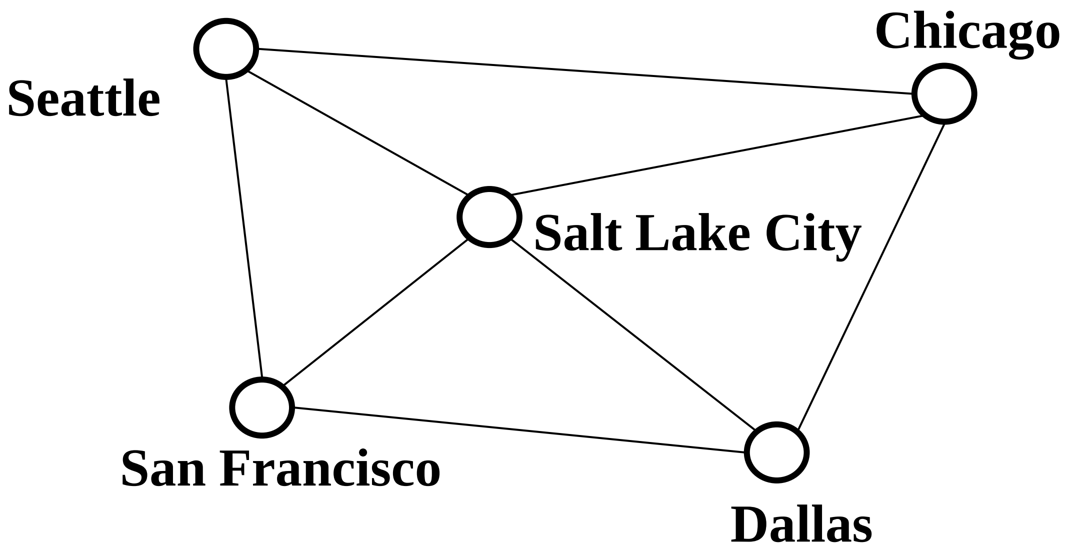
Do negative weights make sense?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps
- Airline routes
- Family trees
- Course pre-requisites
- …

Paths and Cycles

- A path is a list of vertices $[v_0, v_1, \ldots, v_n]$ such that $(v_i, v_{i+1}) \in E$ for all $0 \le i < n$. We say *"a path from* v_0 *to* v_n*"*

- A cycle is a path that begins and ends at the same node ($v_0 == v_n$)



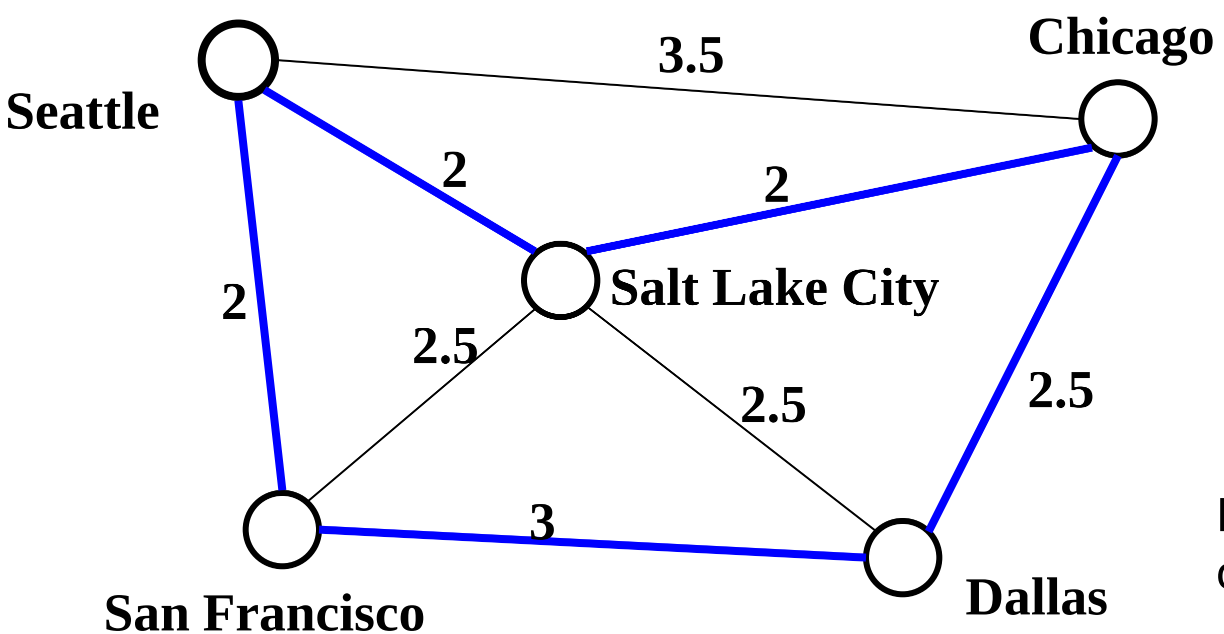
Example path (that also happens to be a cycle):
[Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

Path Length and Cost

- Path length: Number of *edges* in a path
- Path cost: Sum of the weights of each edge

Example where

P= [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]



length(P) = 5

cost(P) = 11.5

Length can sometimes be called “unweighted cost”

Simple Paths and Cycles

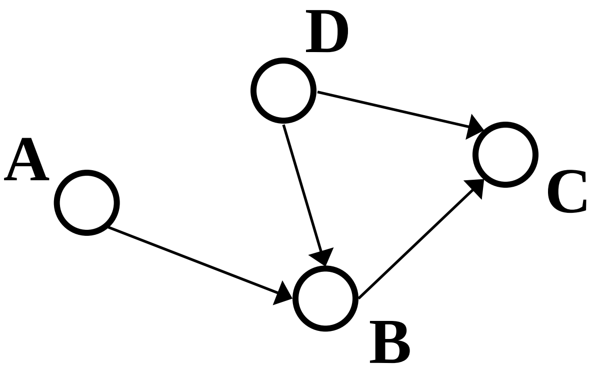
- A simple path repeats no vertices, (except the first might be the last):
  [Seattle, Salt Lake City, San Francisco, Dallas]
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins:
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is a cycle and a simple path:
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

Paths and Cycles in Directed Graphs

Example:

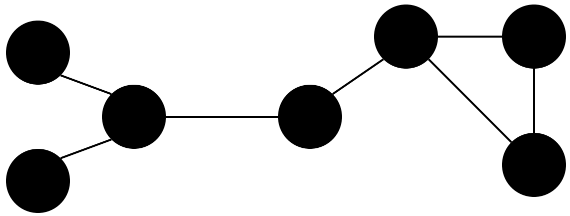


Is there a path from A to D? No

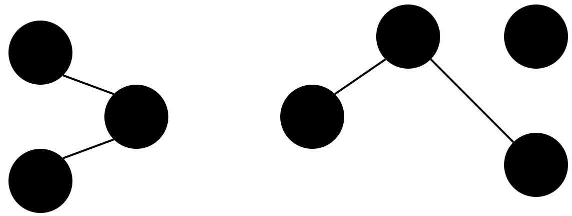
Does the graph contain any cycles? No

Undirected Graph Connectivity

- An undirected graph is connected if for all pairs of vertices **`u,v`**, there exists a *path* from **`u`** to **`v`**

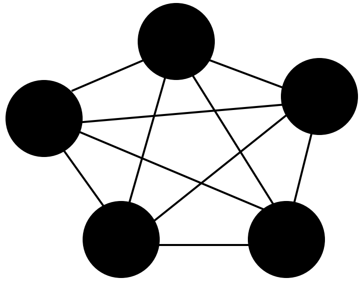


Connected graph



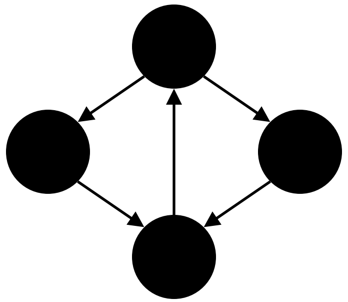
Disconnected graph

- An undirected graph is complete, a.k.a. fully connected, if for all pairs of vertices **`u,v`**, there exists an *edge* from **`u`** to **`v`**

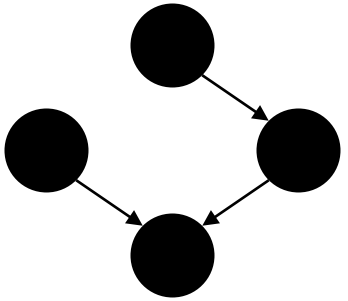


Directed Graph Connectivity

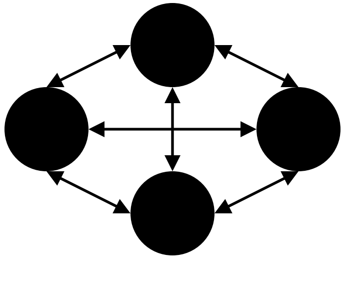
- A directed graph is strongly connected if there is a path from every vertex to every other vertex



- A directed graph is weakly connected if there is a path from every vertex to every other vertex *ignoring direction of edges*



- A direct graph is complete, a.k.a. fully connected, if for all pairs of vertices **`u,v`**, there exists an *edge* from **`u`** to **`v`**



Examples

For undirected graphs: connected?

For directed graphs: strongly connected? weakly connected?

- Web pages with links
- Facebook friends
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
- …

Trees as Graphs

When talking about graphs,

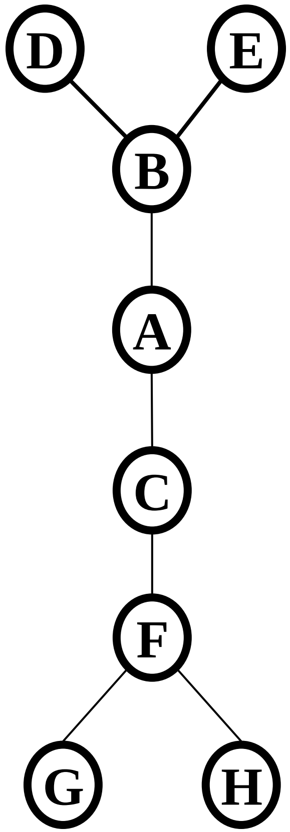
we say a tree is a graph that is:

- undirected
- acyclic
- connected

So all trees are graphs,
but not all graphs are trees

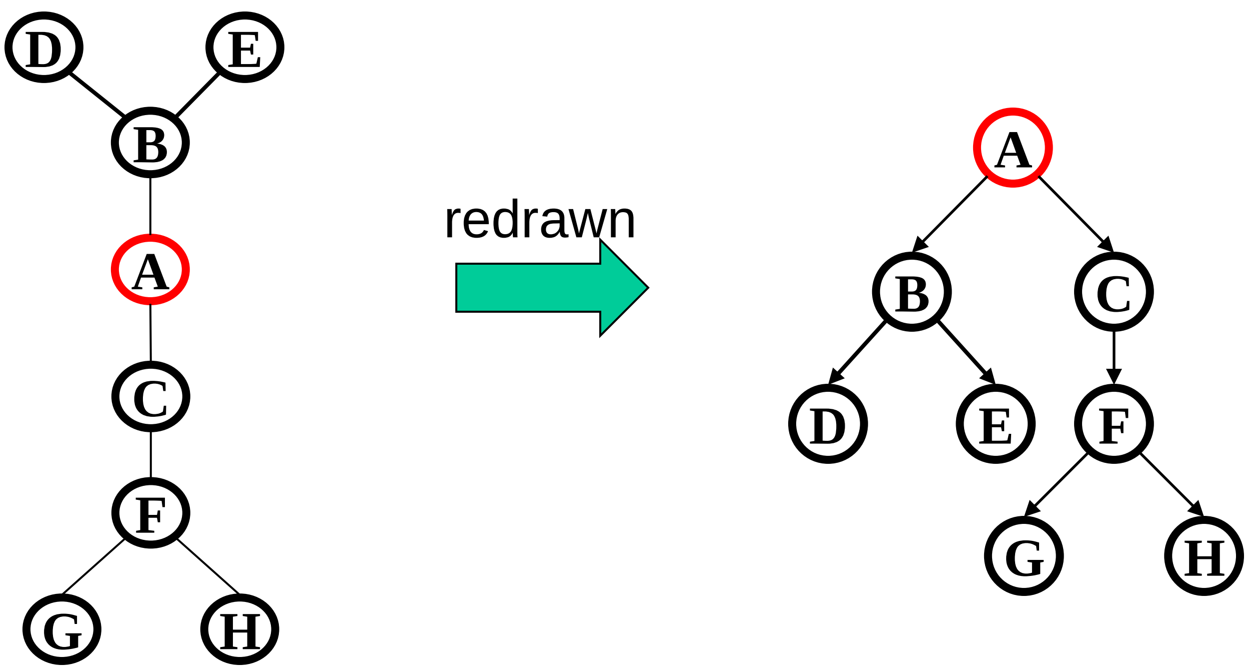
How does this relate to
the trees we know and love?

Example:



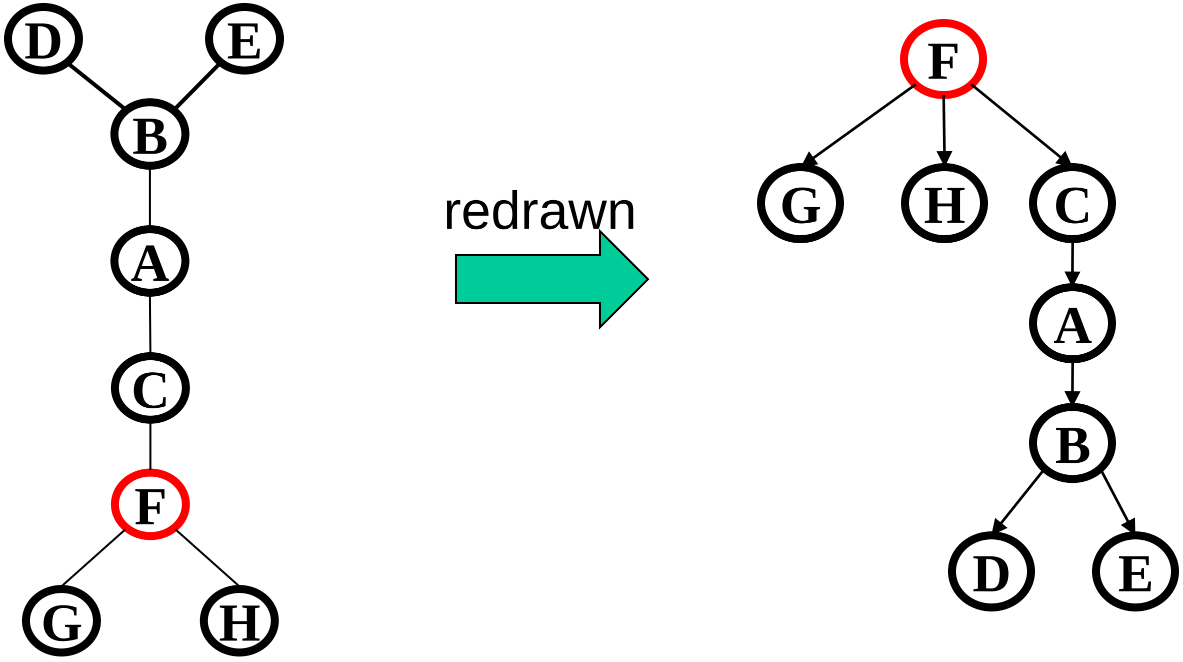
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
  - The tree is simply drawn differently and with undirected edges



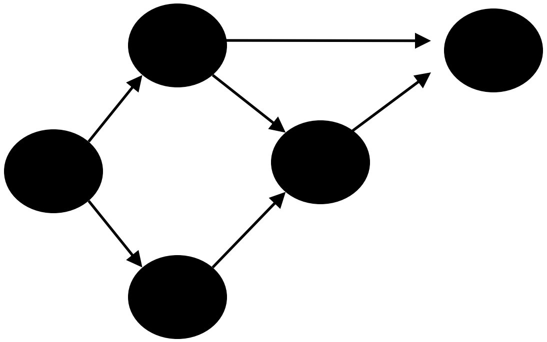
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
  - The tree is simply drawn differently and with undirected edges

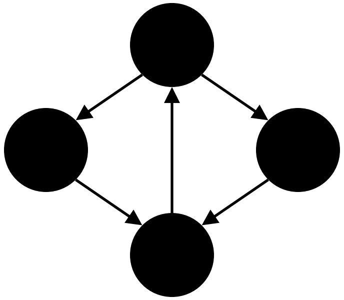


Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph with no directed cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree



  - Every DAG is a directed graph
  - But not every directed graph is a DAG



Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- “Input data” for the Kevin Bacon game
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites
- …

Density / Sparsity

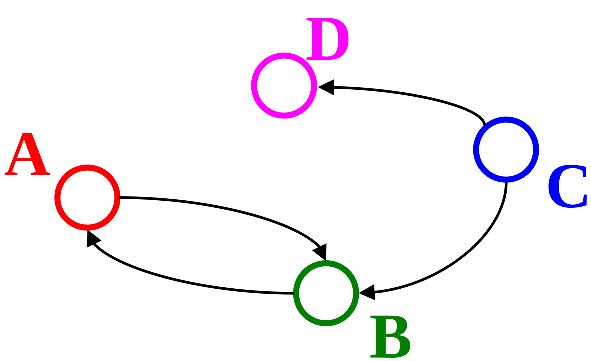
- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph, $0 \leq |E| \leq |V|^2$
- So for any graph, $|E|$ is $O(|V|^2)$
- Another fact: If an undirected graph is *connected*, then $|E| \geq |V|-1$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate as $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight (i.e., $|E|$ is $\Theta(|V|^2)$), we say the graph is dense
    - More sloppily, dense means "lots of edges"
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means "most possible edges missing"

What's the Data Structure?

- So graphs are really useful for lots of data and questions
  - For example, "what's the lowest-cost path from x to y"
- But we need a data structure that represents graphs
- Which data structure is "best" can depend on:
  - properties of the graph
    (e.g., dense versus sparse)
  - the common queries about the graph
    (e.g., "is **`(u,v)`** an edge?" vs "what are the neighbors of node **`u`**?")
- So we will discuss the two standard graph representations
  - Adjacency Matrix and Adjacency List
  - Different trade-offs, particularly time versus space

Adjacency Matrix

- Assign each node a number from **`0`** to **`|V|-1`**
- A |V| x |V| matrix of Booleans (or 0 vs. 1)
  - Then **`M[u][v] == true`** means there is an edge from **`u`** to **`v`**



	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

- Running time to:
  - Get a vertex's out-edges:
  - Get a vertex's in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:

- Space requirements:

- Best for sparse or dense graphs?

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

Adjacency Matrix Properties

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

- Running time to:
  - Get a vertex's out-edges: $O(|V|)$
  - Get a vertex's in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- Space requirements:
  - $|V|^2$ bits

- Best for sparse or dense graphs?
  - Best for dense graphs

Adjacency Matrix Properties

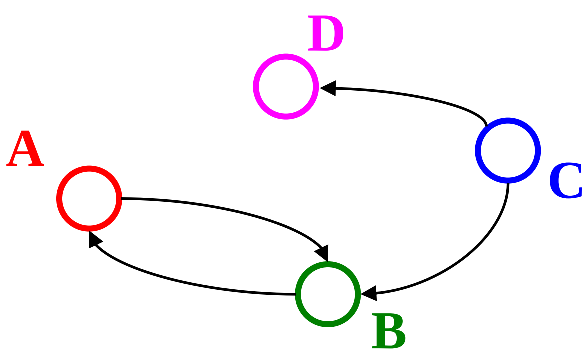
- How will the adjacency matrix vary for an **undirected graph**?
  - Undirected will be symmetric about diagonal axis

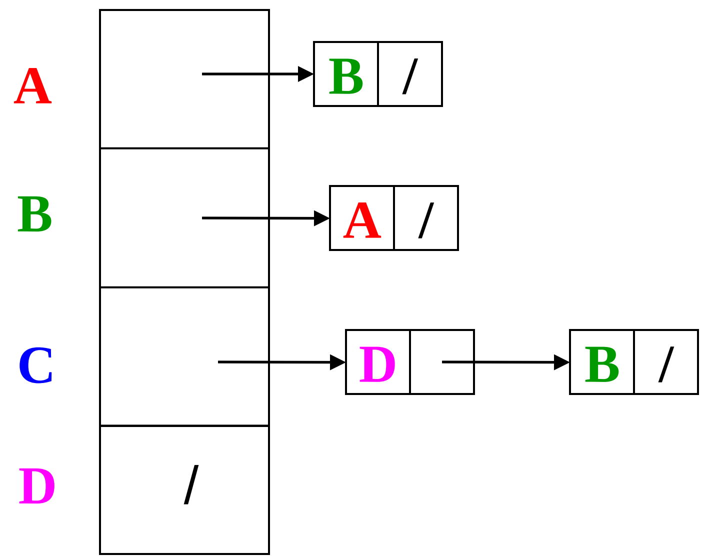
- How can we adapt the representation for **weighted graphs**?
  - Instead of a Boolean, store an number in each cell
  - Need some value to represent ‘not an edge’
    - 0, -1, or some other value based on how you are using the graph

	A	B	C	D
A	F	T	F	F
B	T	F	F	F
C	F	T	F	T
D	F	F	F	F

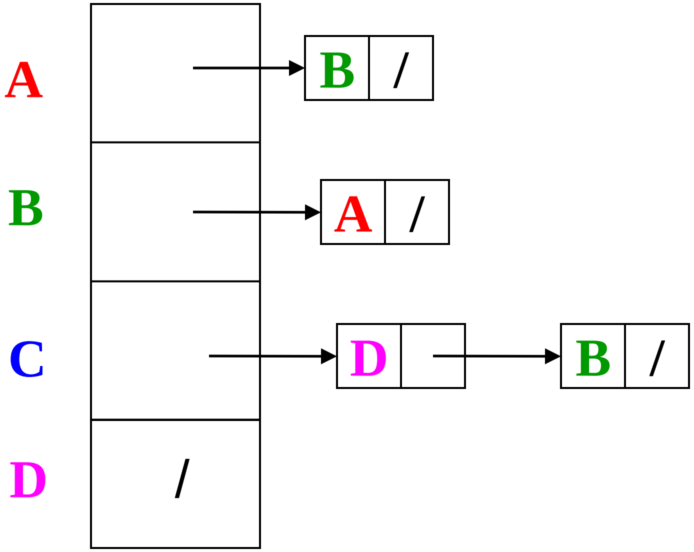
Adjacency List

- Assign each node a number from **`0`** to **`|V|-1`**
- An array of length **`|V|`** in which each entry stores a list of all adjacent vertices (e.g., linked list)



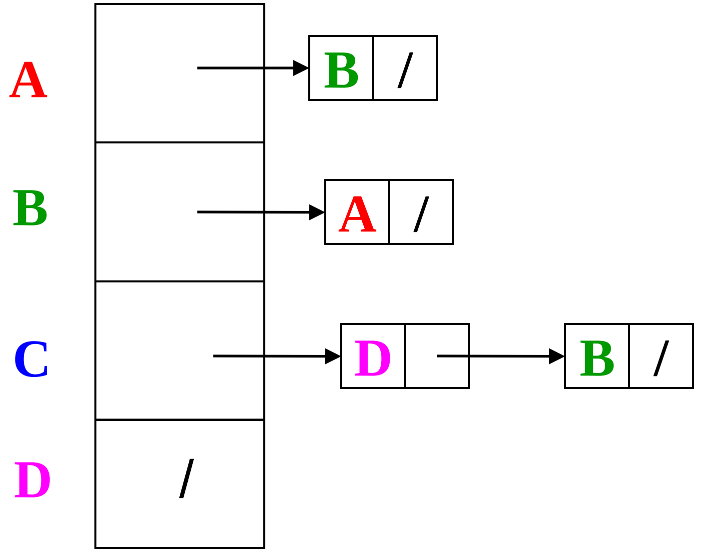


Adjacency List Properties



- Running time to:
  - Get all of a vertex's out-edges:
  - Get all of a vertex's in-edges:
  - Decide if some edge exists:
  - Insert an edge:
  - Delete an edge:
- Space requirements:
  -
- Best for dense or sparse graphs?
  -

Adjacency List Properties



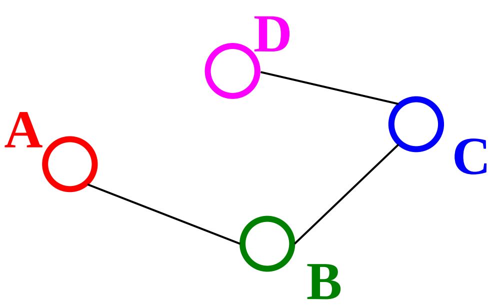
- Running time to:
  - Get all of a vertex's out-edges:
    O(*d*) where *d* is out-degree of vertex
  - Get all of a vertex's in-edges:
    O(|E|) (but could keep a second adjacency list for this!)
  - Decide if some edge exists:
    O(*d*) where *d* is out-degree of source
  - Insert an edge: *O*(1) (unless you need to check if it's there)
  - Delete an edge: *O*(*d*) where *d* is out-degree of source
- Space requirements:
  - *O*(|V|+|E|)
- Best for dense or sparse graphs?
  - Best for sparse graphs, so usually just stick with linked lists

Undirected Graphs

Adjacency matrices & adjacency lists both do fine for undirected graphs

- Matrix: Could save space by using only about half the array
  - How would you “get all neighbors”?
- Lists: Each edge in two lists to support efficient “get all neighbors”

Example:



	A	B	C	D
A	F	T	F	F
B	T	F	T	F
C	F	T	F	T
D	F	F	T	F