CSE332: Data Abstractions

Lecture 10: Comparison Sorting

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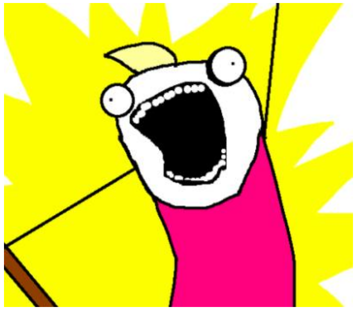
Introduction to Sorting

- We have covered stacks, queues, priority queues, and dictionaries
    - All focused on providing one element at a time

- But often we know we want “all the things” in some order
    - Anyone can sort, but a computer can sort faster
    - Very common to need data sorted somehow
        - Alphabetical list of people
        - List of countries ordered by population



- Algorithms have different asymptotic and constant-factor trade-offs
    - No single “best” sort for all scenarios
    - Knowing “one way to sort” is not sufficient

More Reasons to Sort

General technique in computing:

Preprocess data to make subsequent operations faster

Example: Sort the data so that you can

- Find the **`k`**th largest in constant time for any **`k`**
- Perform binary search to find elements in logarithmic time

Whether the performance of the preprocessing matters depends on

- How often the data will change
- How much data there is

Careful Statement of the Basic Problem

Assume we have *n* comparable elements in an array,
and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record (potentially a set of fields)
- A comparison function (must be consistent and total)
    - Given keys a and b, what is their relative ordering? <, =, >?

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**,
  if **`i < j`** then **`A[i] ≤ A[j]`**
- Unspoken assumption: **`A`** must have all the data it started with

An algorithm doing this is a comparison sort

Variations on the basic problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms need not do so)

2. Maybe ties need to be resolved by “original array position”
   - Sorts that do this naturally are called stable sorts
   - Others could tag each item with its original position and adjust their comparisons (non-trivial constant factors)

3. Maybe we must not use more than *O*(1) “auxiliary space”
   - Sorts meeting this requirement are called in-place sorts

4. Maybe we can do more with elements than just compare
   - Sometimes leads to faster algorithms

5. Maybe we have too much data to fit in memory
   - Use an “external sorting” algorithm

Sorting: The Big Picture

Simple algorithms: O(n^2)

- Insertion sort
- Selection sort
- Shell sort
- …

Fancier algorithms: O($n \log n$)

- Heap sort
- Merge sort
- Quick sort (avg)
- …

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: O(n)

- Bucket sort
- Radix sort

Handling huge data sets

- External sorting

Insertion Sort

- Idea: At step **`k`**,
  put the **`k`**th input element in the correct position
  among the first **`k`** elements

- Alternate way of saying this:
  - Sort first element (this is easy)
  - Now insert 2nd element in order
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted

- Time?

  Best-case _____ Worst-case _____ "Average" case ____

Insertion Sort

- Idea: At step **`k`**,
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  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted

- Time?

Best-case	O(n)	Worst-case	$O(n^2)$	"Average" case	$O(n^2)$
start sorted		start reverse sorted		(see text)	

Selection Sort

- Idea: At step `k`,
  find the smallest element among the unsorted elements
  and put it at position k

- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …

- “Loop invariant”: when loop index is `i`,
  first `i` elements are the `i` smallest elements in sorted order

- Time?

  Best-case _____ Worst-case _____ “Average” case ____

Selection Sort

- Idea: At step **`k`**,
  find the smallest element among the unsorted elements
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- Alternate way of saying this:
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  - Find next smallest element, put it 3rd
  - …

- "Loop invariant": when loop index is **`i`**,
  first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case $O(n^2)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$

  Always $T(1) = 1$ and $T(n) = n + T(n-1)$

Mystery Sort

This is one implementation of which sorting algorithm (shown for ints)?

```
void mystery(int[] arr) {
  for(int i = 1; i < arr.length; i++) {
     int tmp = arr[i];
     int j;
     for(j=i; j > 0 && tmp < arr[j-1]; j--)
        arr[j] = arr[j-1];
     arr[j] = tmp;
  }
}
```

Note: As with heaps, “moving the hole” is faster than unnecessary swapping (impacts constant factor)

Insertion Sort vs. Selection Sort

- They are different algorithms

- They solve the same problem

- Have the same worst-case and average-case asymptotic complexity
    - Insertion-sort has better best-case complexity;
      preferable when input is "mostly sorted"

- Other algorithms are more efficient
  for non-small arrays that are not already almost sorted
    - Small arrays may do well with Insertion sort

Aside: We Will Not Cover Bubble Sort

- It does not have good asymptotic complexity: $O(n^2)$
- It is not particularly efficient with respect to constant factors
- Almost everything it is good at,
  some other algorithm is at least as good at
- Perhaps some people teach it just because it was taught to them

- For fun see: “Bubble Sort: An Archaeological Algorithmic Analysis”, Owen Astrachan, SIGCSE 2003

Sorting: The Big Picture

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- Insertion sort
- Selection sort
- Shell sort
- …

Fancier algorithms: O(n log n)

- Heap sort
- Merge sort
- Quick sort (avg)
- …

Comparison lower bound: Ω(n log n)

Specialized algorithms: O(n)

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Handling huge data sets

- External sorting

Heap Sort

- As you are seeing in Project 2, sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, or better yet do a **`buildHeap`**
  - **`for(i=0; i < arr.length; i++)`**

    `arr[i] = deleteMin();`

- Worst-case running time: **$O(n \log n)$**

  Why?

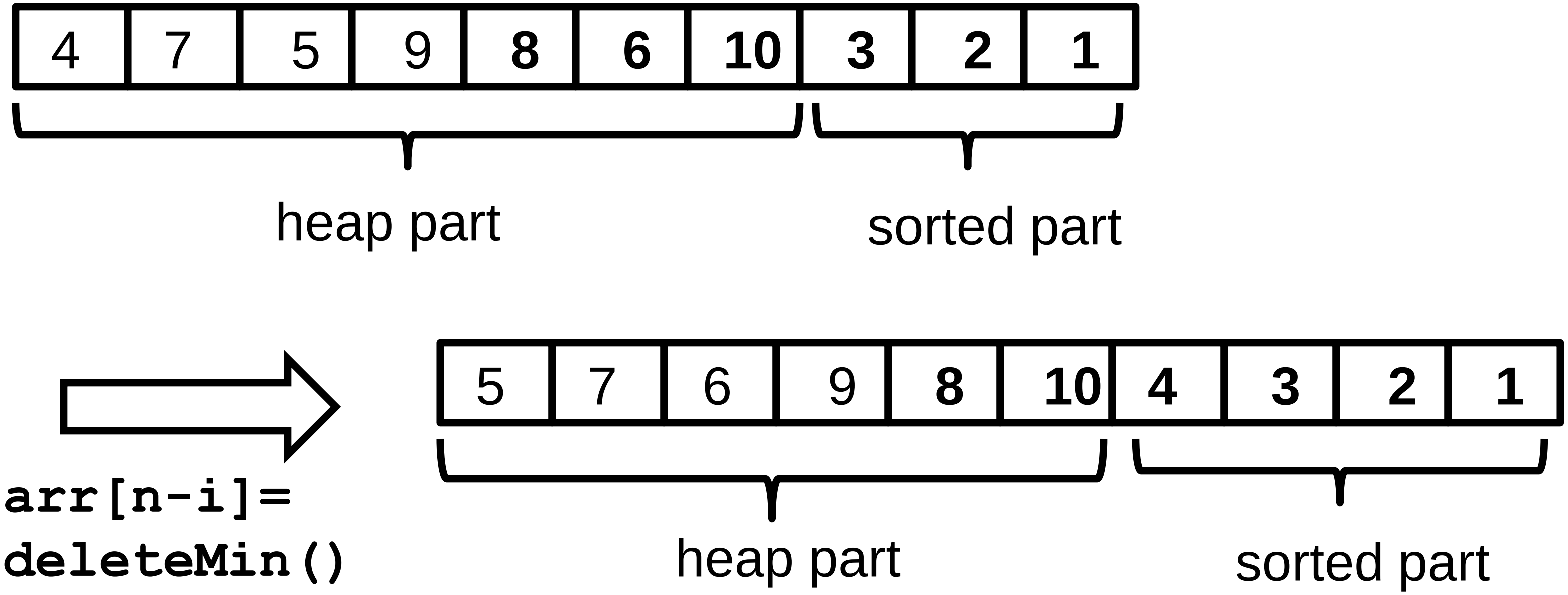
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place

In-Place Heap Sort

But this reverse sorts – how would you fix that?

Reverse your comparator, so you build a maxHeap

- Treat the initial array as a heap (via **`buildHeap`**)
- When you delete the **`i`**th element, put it at **`arr[n-i]`**
  - That array location is not part of the heap anymore!



4	7	5	9	**8**	**6**	**10**	**3**	**2**	**1**

heap part

sorted part

`arr[n-i]=`
`deleteMin()`

5	7	6	9	**8**	**10**	**4**	**3**	**2**	**1**

heap part

sorted part

"AVL sort"

- We can also use a balanced tree to:
  - **`insert`** each element: total time $O(n \log n)$
  - Repeatedly **`deleteMin`**: total time $O(n \log n)$

- But this cannot be made in-place,
  and it has worse constant factors than heap sort
  - both are $O(n \log n)$ in worst, best, and average case
  - neither parallelizes well
  - heap sort is better

- Do not even think about trying to sort with a hash table

Divide and Conquer

Very important technique in algorithm design

1. Divide problem into smaller parts
2. Independently solve the simpler parts
    - Think recursion
    - Or potential parallelism
3. Combine solution of parts to produce overall solution

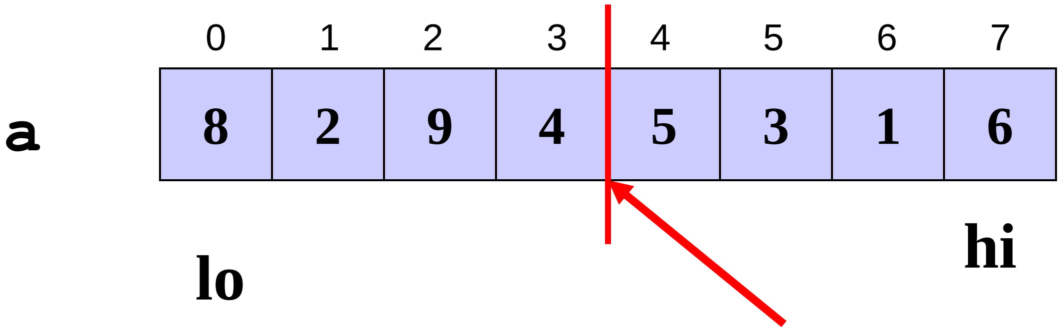
Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole

2. Quicksort: Pick a “pivot” element
   Divide elements into less-than pivot
   and greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is [*sorted-less-than,*
   then *pivot,*
   then *sorted-greater-than*]

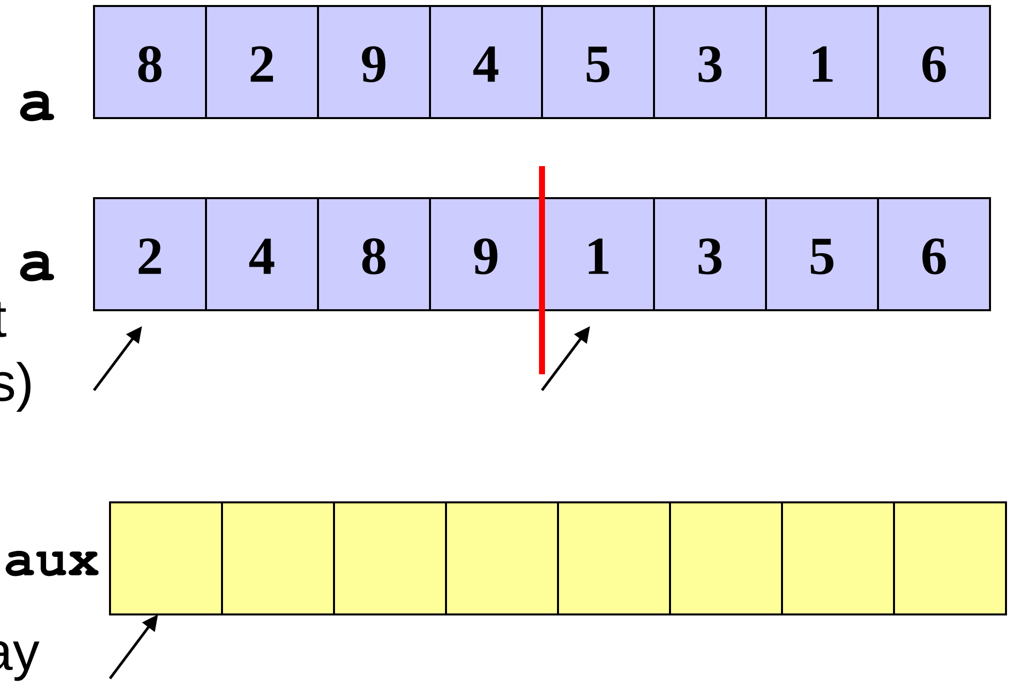
Mergesort



- To sort array from position `lo` to position `hi`:
  - If range is 1 element long, it is already sorted! (our base case)
  - Else, split into two halves:
    - Sort from `lo` to `(hi+lo)/2`
    - Sort from `(hi+lo)/2` to `hi`
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - *O*(*n*) but requires auxiliary space…

Example: Focus on Merging



Start with:

`a`

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

`a`

2	4	8	9	1	3	5	6

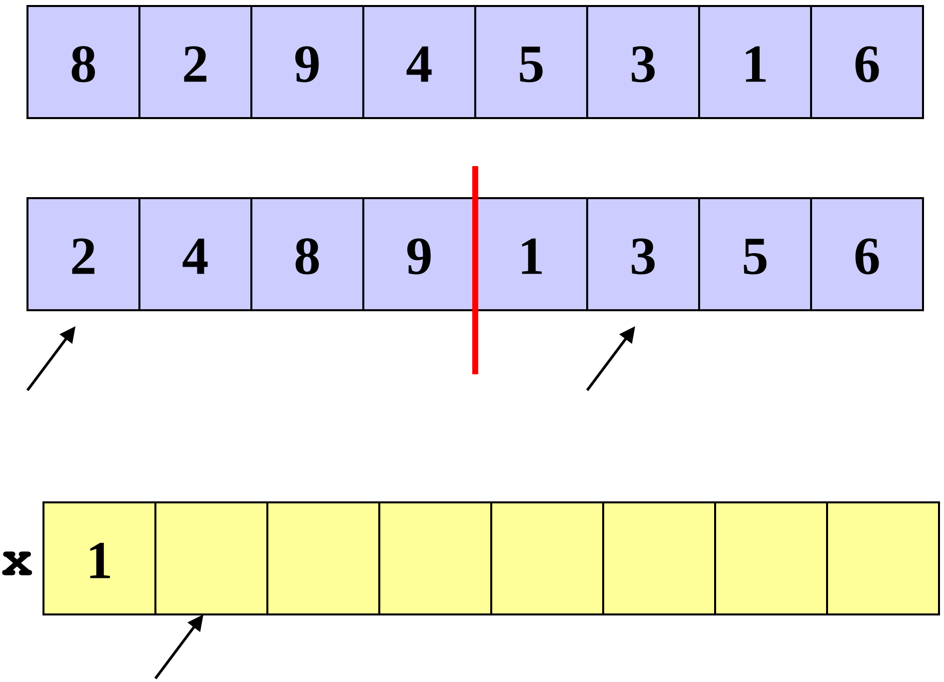
Merge:

Use 3 “fingers” and 1 more array

`aux`

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

a

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

a

2	4	8	9	1	3	5	6

Merge:

Use 3 “fingers”

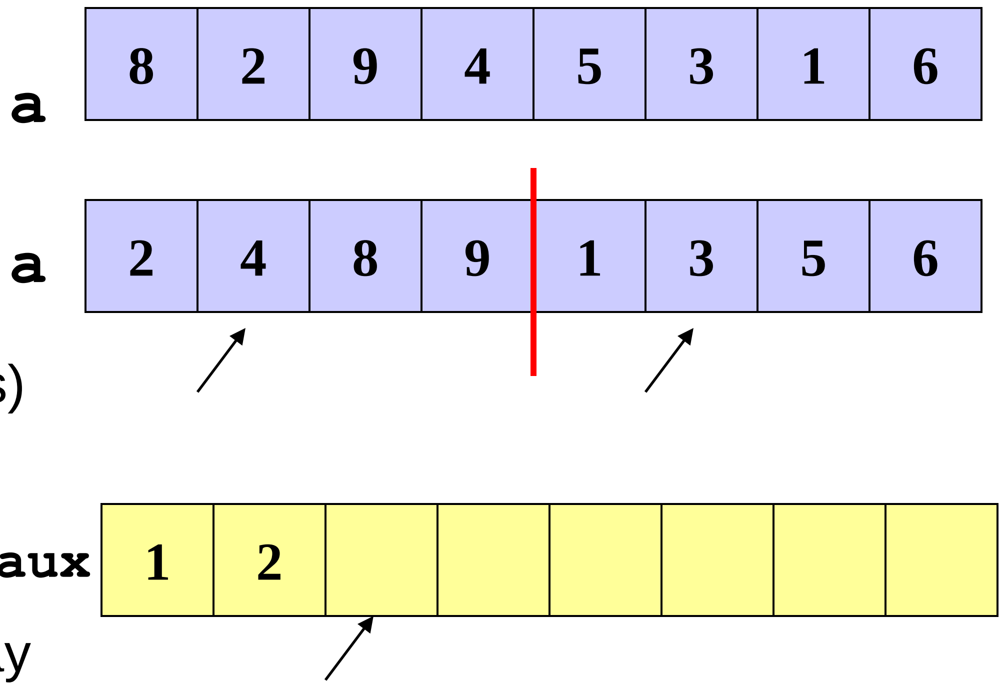
and 1 more array

aux

1							

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

`a`	8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

`a`	2	4	8	9	1	3	5	6

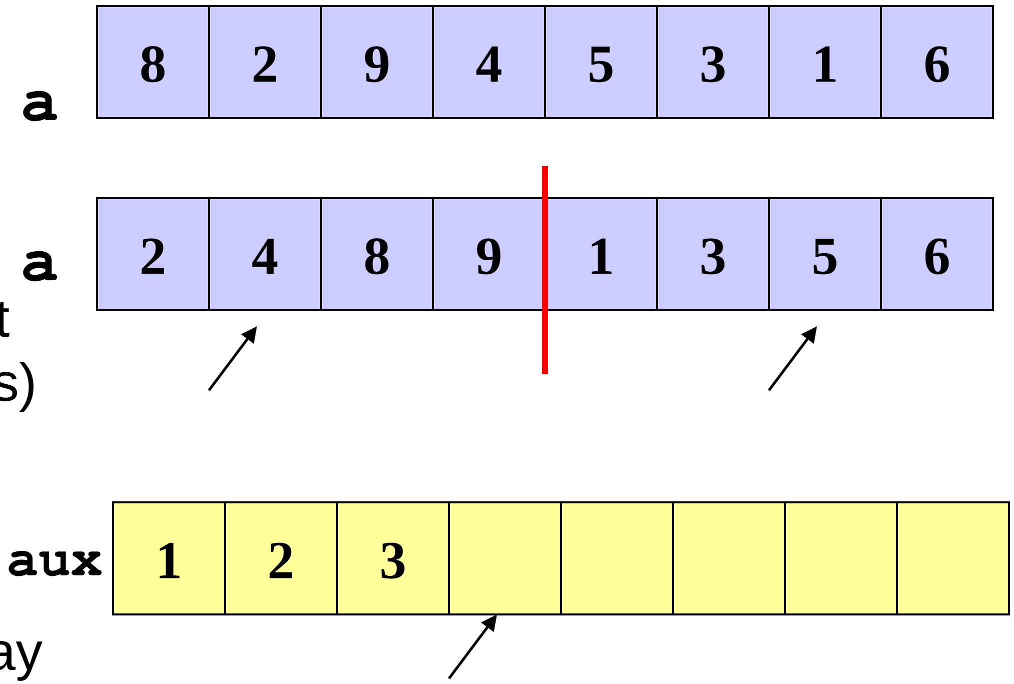
Merge:

Use 3 “fingers” and 1 more array

`aux`	1	2						

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

a

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

a

2	4	8	9	1	3	5	6

Merge:

Use 3 “fingers”

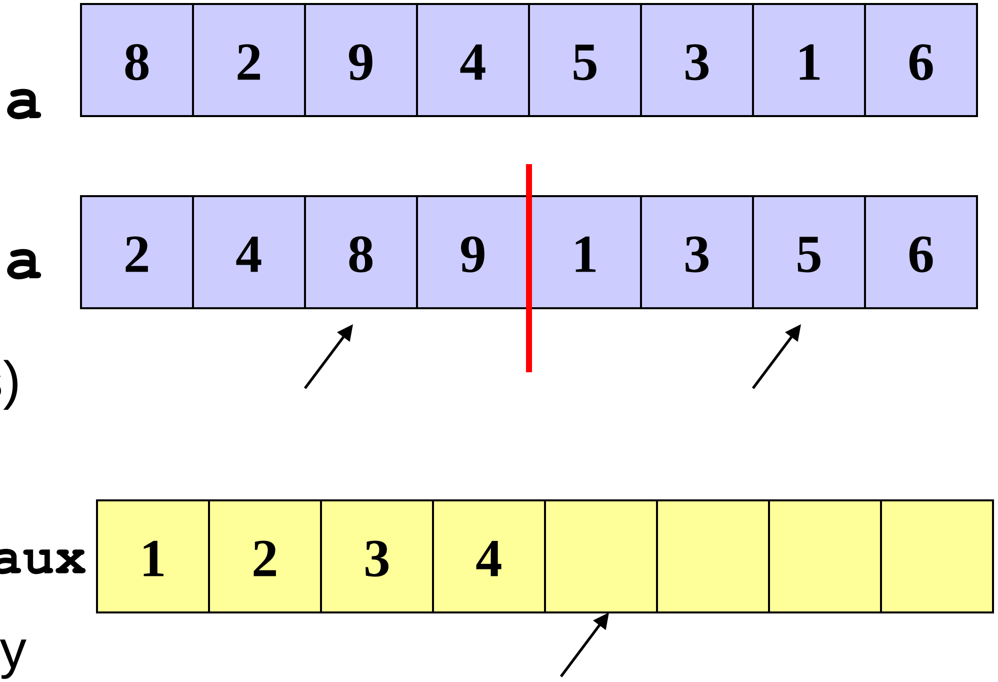
and 1 more array

aux

1	2	3					

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

`a`	8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

`a`	2	4	8	9	1	3	5	6

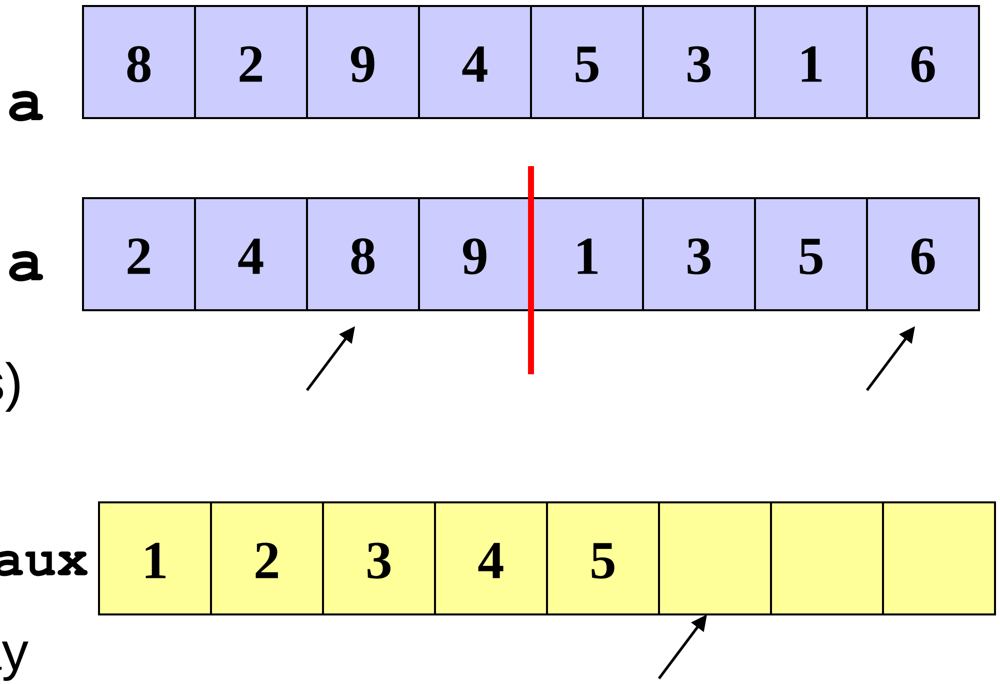
Merge:

Use 3 “fingers” and 1 more array

`aux`	1	2	3	4				

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

After recursion:

(for now we just
assume it works)

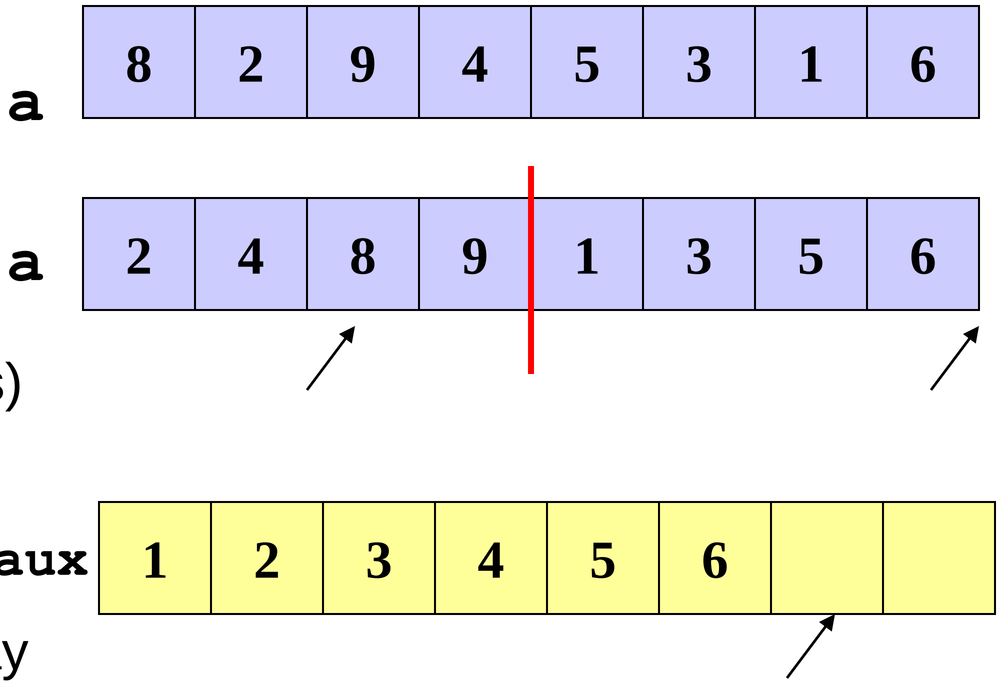
Merge:

Use 3 “fingers”

and 1 more array

(After merge,
copy back to
original array)

Example: Focus on Merging



Start with:

a	8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

a	2	4	8	9	1	3	5	6

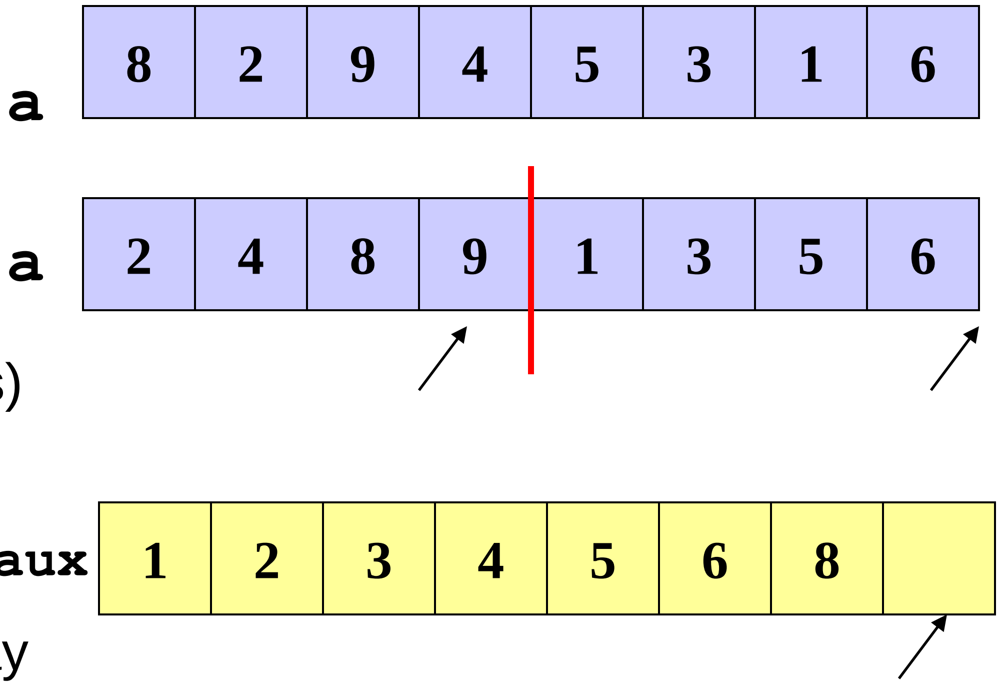
Merge:

Use 3 “fingers” and 1 more array

aux	1	2	3	4	5	6		

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

a

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

a

2	4	8	9	1	3	5	6

Merge:

Use 3 "fingers"

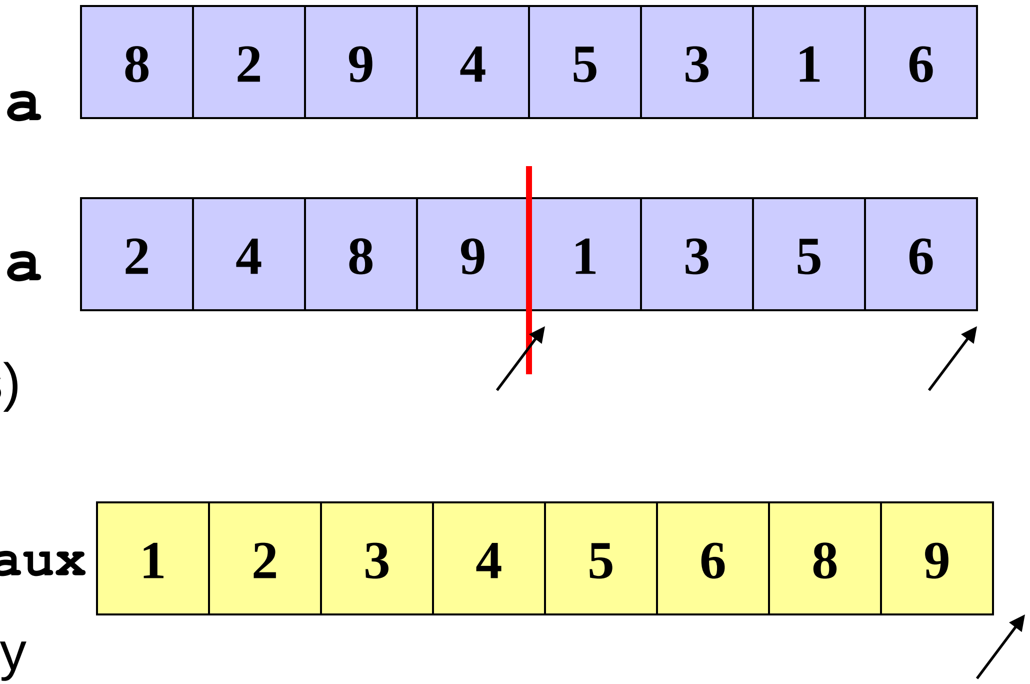
and 1 more array

aux

1	2	3	4	5	6	8	

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

`a`

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

`a`

2	4	8	9	1	3	5	6

Merge:

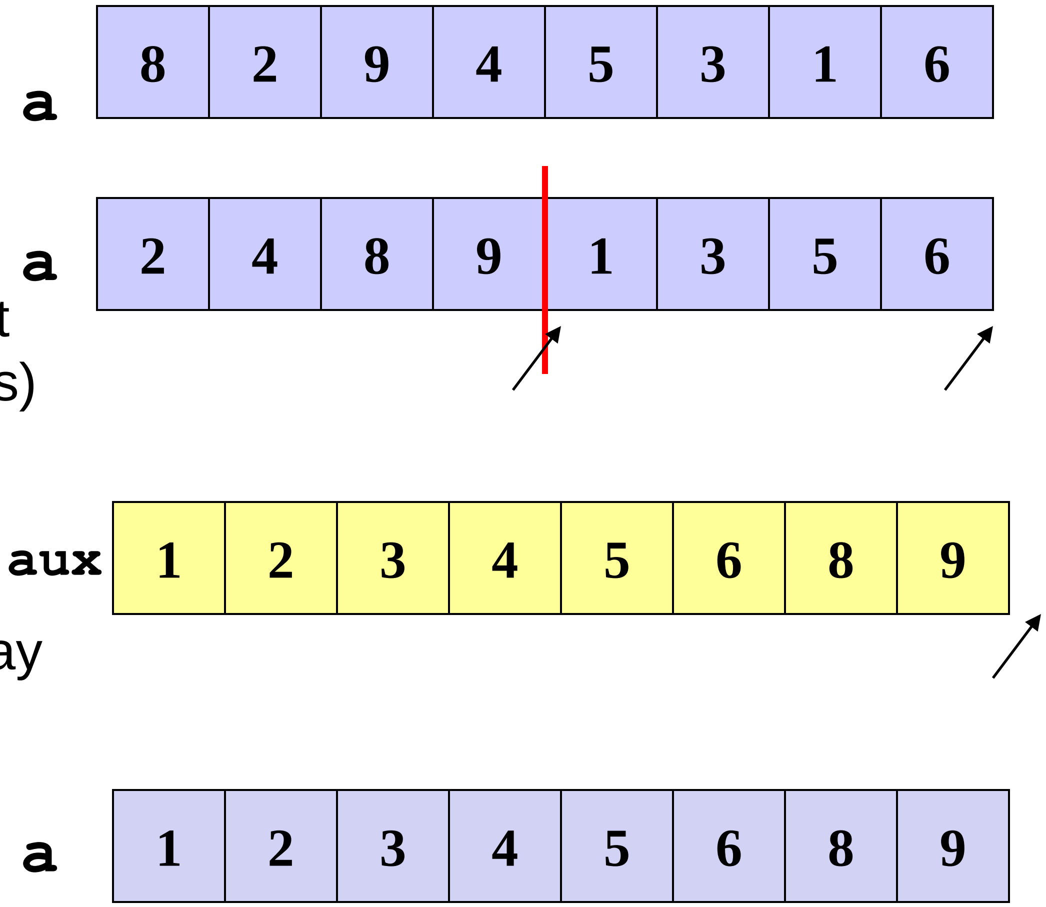
Use 3 “fingers” and 1 more array

`aux`

1	2	3	4	5	6	8	9

(After merge, copy back to original array)

Example: Focus on Merging



Start with:

a

8	2	9	4	5	3	1	6

After recursion:

(for now we just assume it works)

a

2	4	8	9	1	3	5	6

Merge:

Use 3 “fingers” and 1 more array

aux

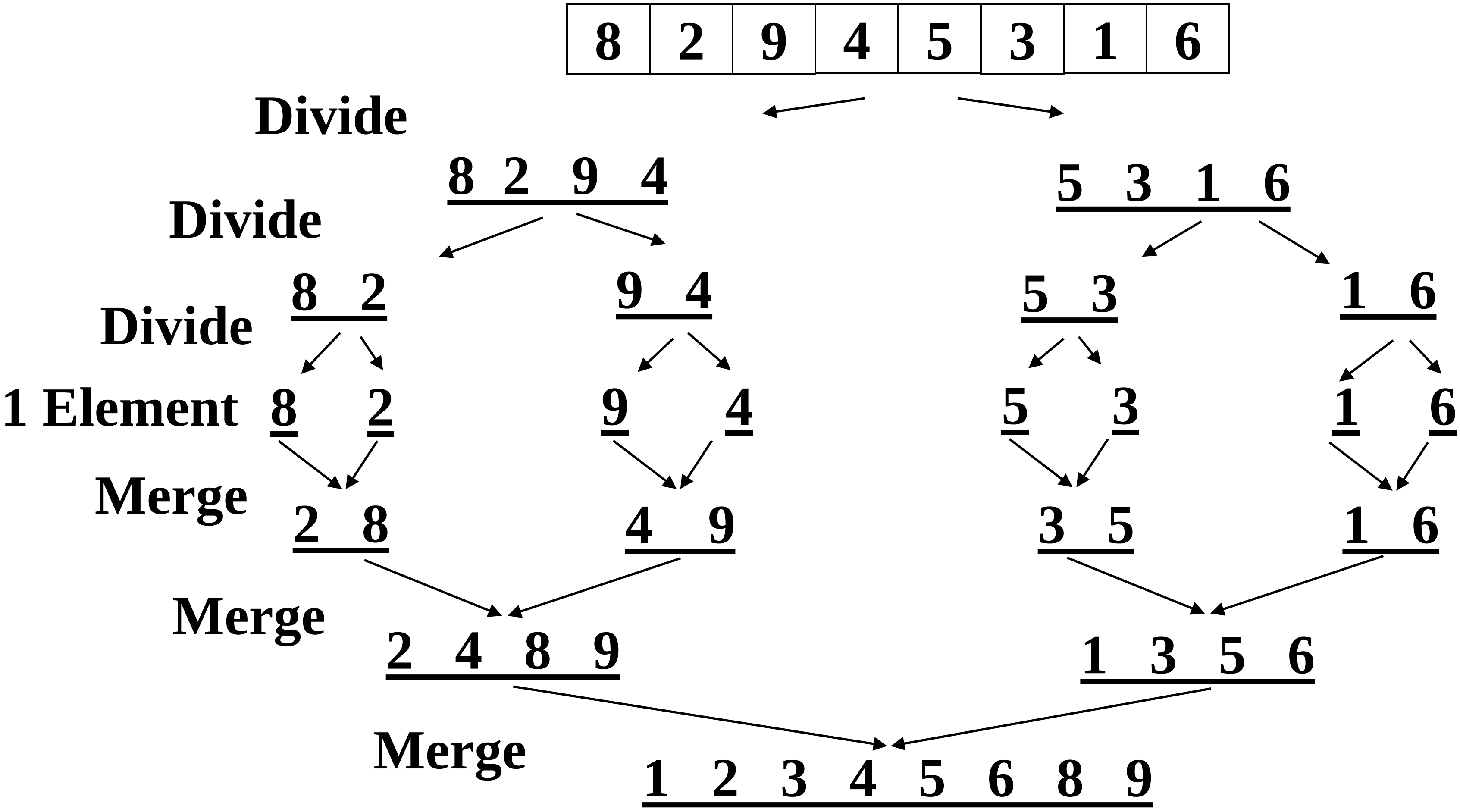
1	2	3	4	5	6	8	9

(After merge, copy back to original array)

a

1	2	3	4	5	6	8	9

Example: Mergesort Recursion



8	2	9	4	5	3	1	6

Divide

8 2 9 4 **5 3 1 6**

Divide

8 2 **9 4** **5 3** **1 6**

Divide

1 Element **8** **2** **9** **4** **5** **3** **1** **6**

Merge

2 8 **4 9** **3 5** **1 6**

Merge

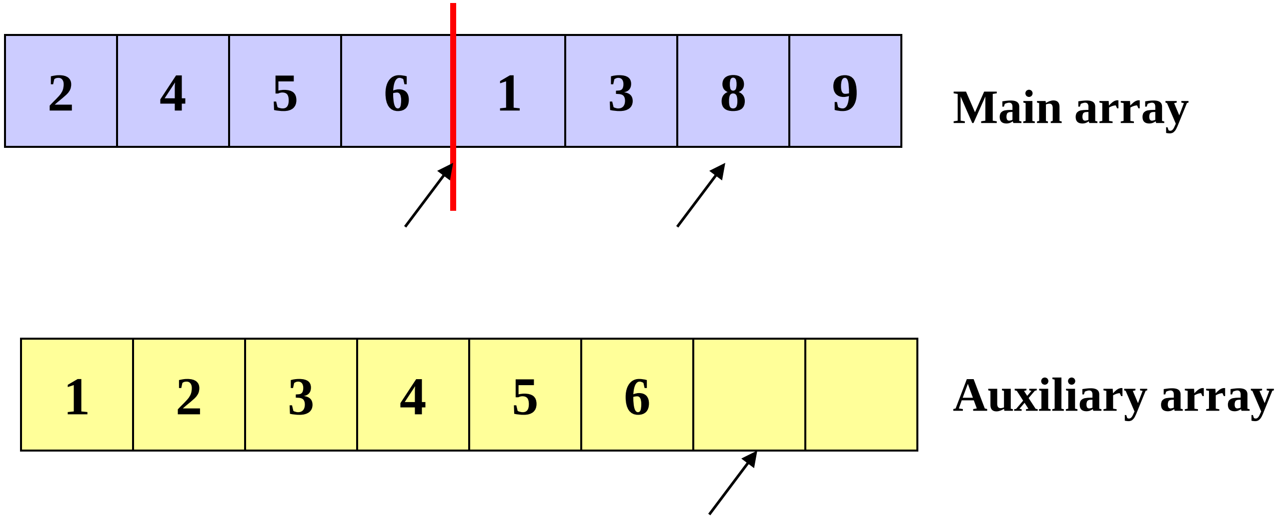
2 4 8 9 **1 3 5 6**

Merge

1 2 3 4 5 6 8 9

Mergesort: Some Time Saving Details

- What if the final steps of our merge looked like this:



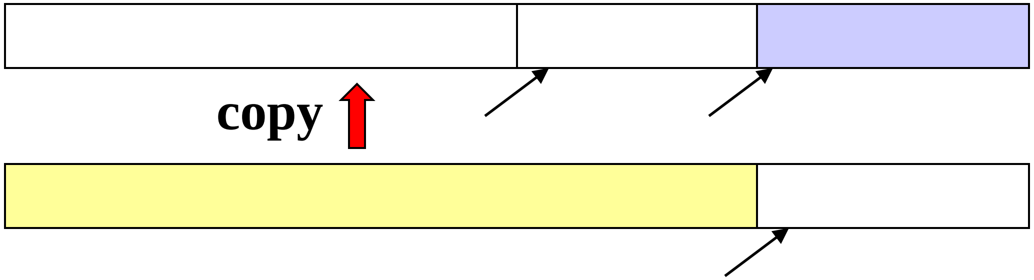
2	4	5	6	1	3	8	9	**Main array**

1	2	3	4	5	6			**Auxiliary array**

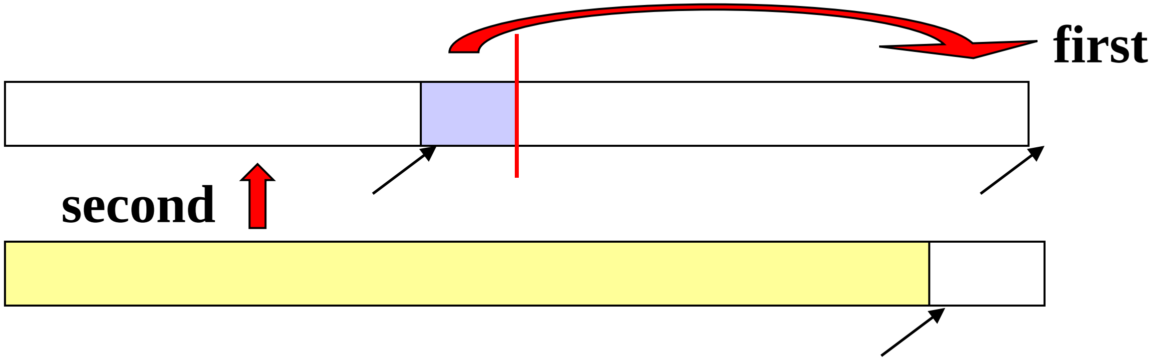
- Wasteful to copy to the auxiliary array just to copy back…

Mergesort: Some Time Saving Details

- If left-side finishes first, just stop the merge and copy back:



- If right-side finishes first, copy dregs into right then copy back:



Mergesort: Saving Space and Copying

Simplest / Worst:

Use a new auxiliary array of size **`(hi-lo)`** for every merge

Better:

Use a new auxiliary array of size **`n`** for every merging stage

Better:

Reuse same auxiliary array of size **`n`** for every merging stage

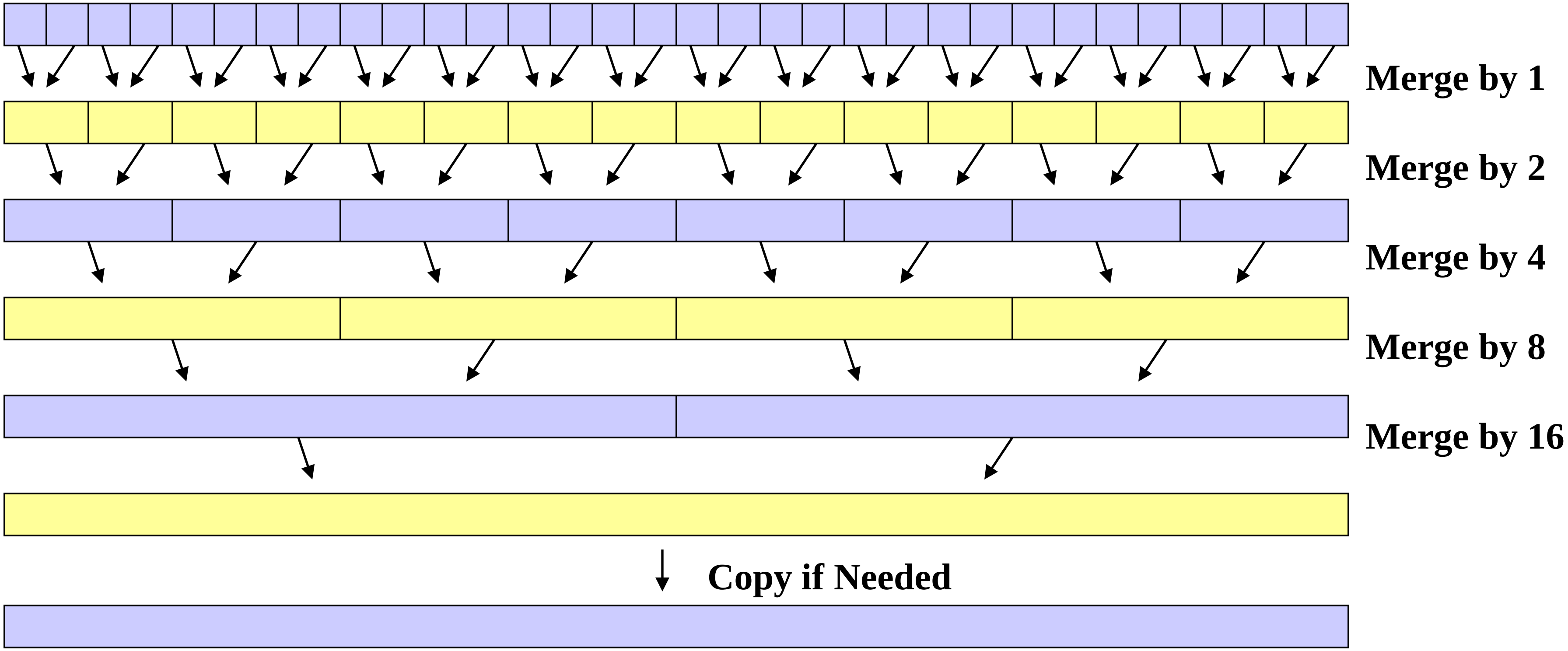
Best:

Do not copy back after merge, instead swap usage of the original and auxiliary array (i.e., even levels move to auxiliary array, odd levels move back to original array)

- Need one copy at end if number of stages is odd

Swapping Original and Auxiliary Array

- First recurse down to lists of size 1
- As we return from the recursion, swap between arrays



- Arguably easier to code without using recursion at all

Mergesort Analysis

Having defined an algorithm and argued it is correct,
we can analyze its running time and space:

To sort *n* elements, we:

- Return immediately if $n=1$
- Else do 2 subproblems of size $n/2$ and then an $O(n)$ merge

Recurrence relation:

$T(1) = c_1$

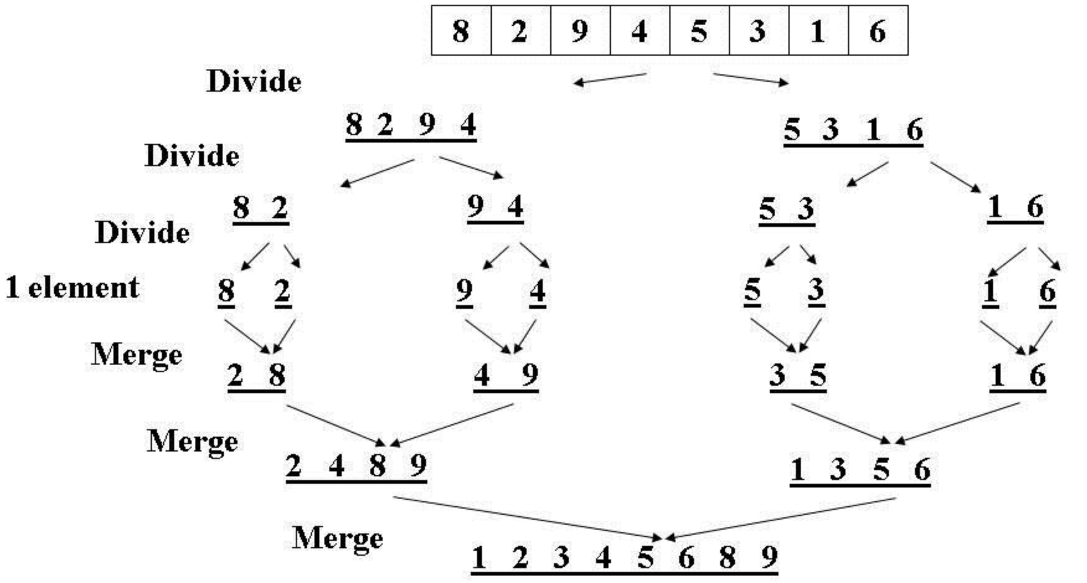
$T(n) = 2T(n/2) + c_2 n$

Mergesort Analysis

This recurrence is common enough you just "know" it's *O*(*n* `log` *n*)

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion "tree" will have `log` *n* height
- At each level we do a *total* amount of merging equal to *n*



Quicksort

- Also uses divide-and-conquer
  - Recursively chop into halves
  - Instead of doing all the work as we merge together, we will do all the work as we recursively split into halves
  - Unlike MergeSort, does not need auxiliary space
- $O(n \log n)$ on average, but $O(n^2)$ worst-case
  - MergeSort is always $O(n \log n)$
  - So why use QuickSort at all?
- Can be faster than Mergesort
  - Believed by many to be faster
  - Quicksort does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

Quicksort Overview

1. Pick a pivot element

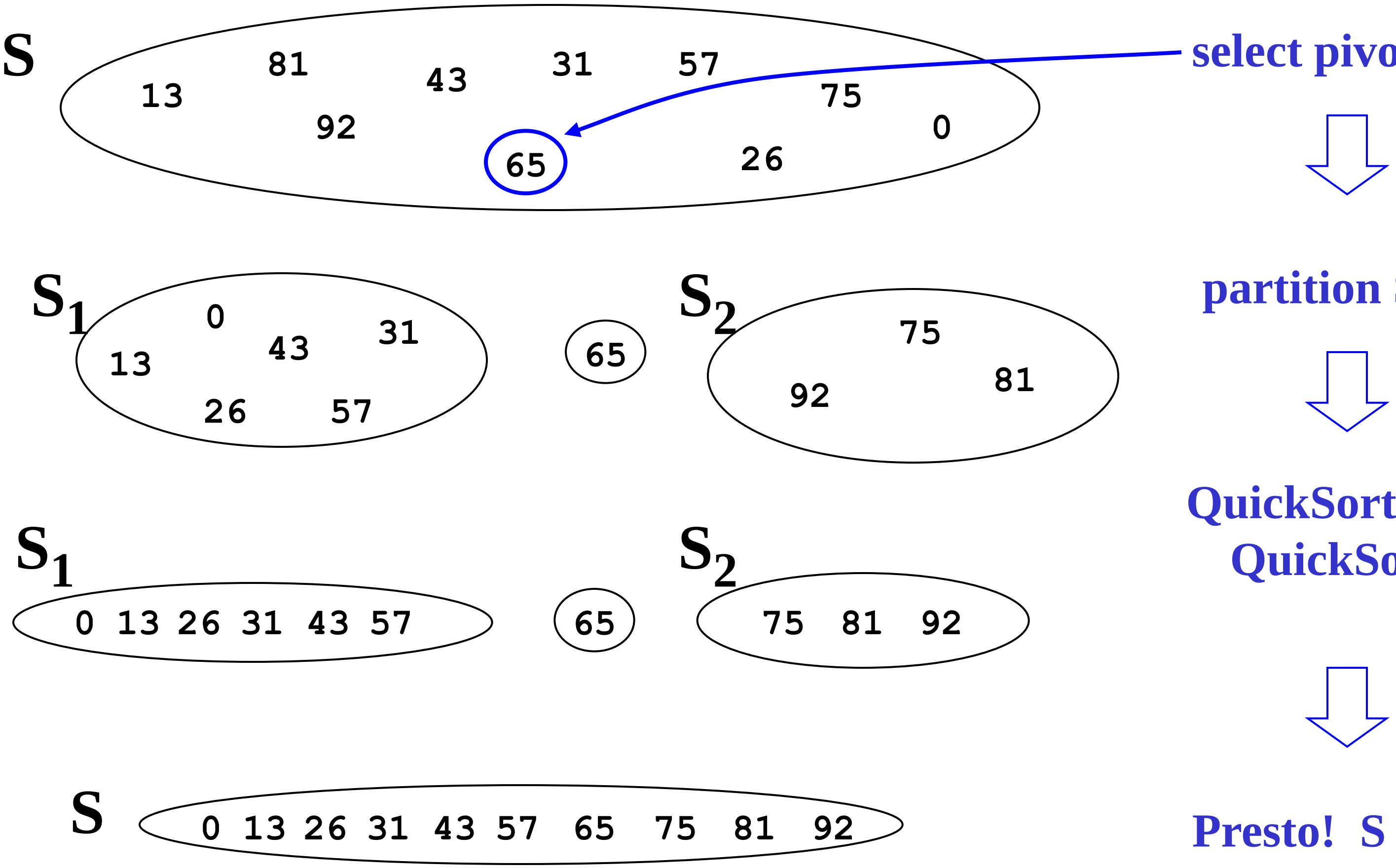
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The answer is as simple as “A, B, C”

Alas, there are some details lurking in this algorithm

Quicksort: Think in Terms of Sets



select pivot value

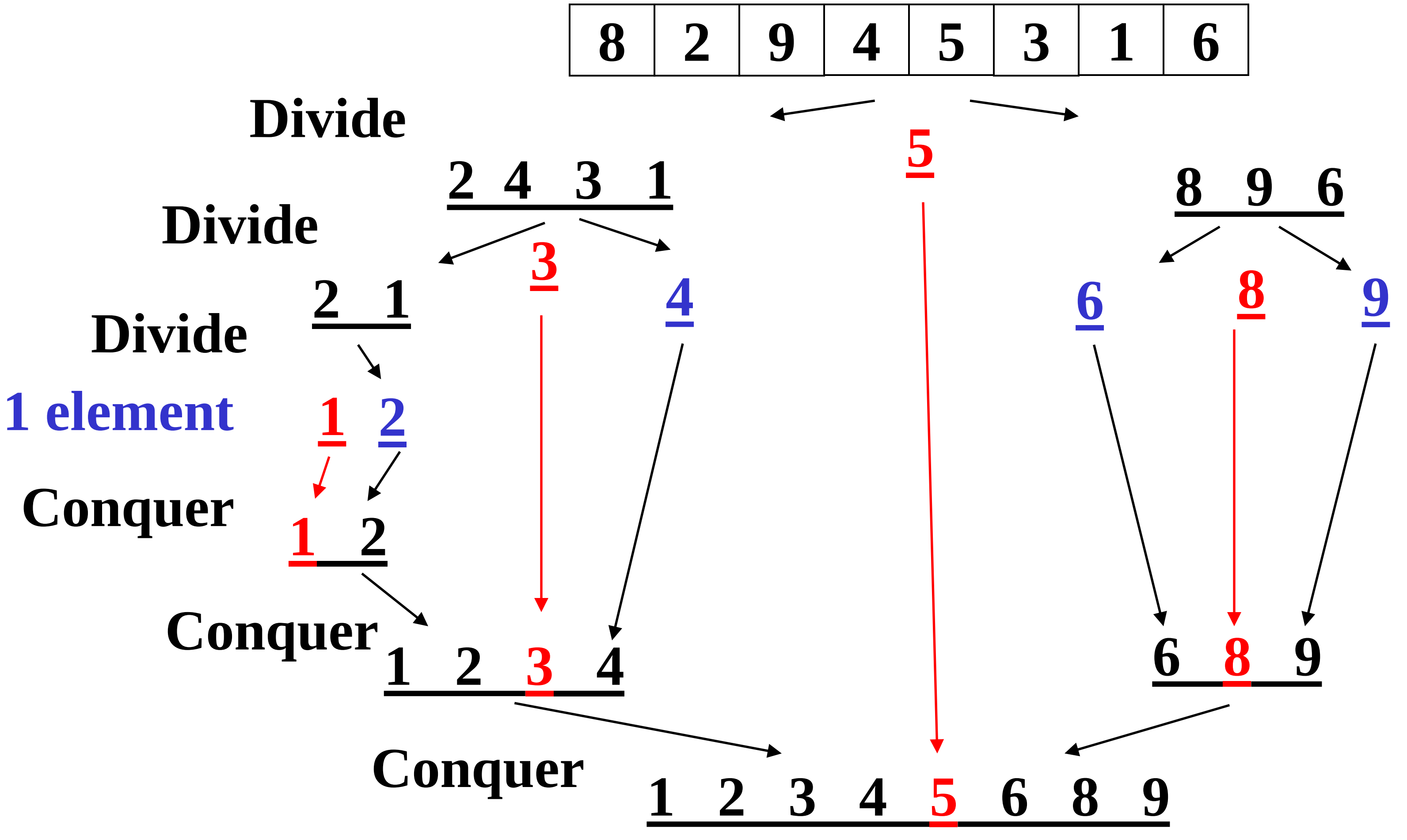
partition S

QuickSort(S_1) and QuickSort(S_2)

Presto! S is sorted

[Weiss]

Example: Quicksort Recursion



8	2	9	4	5	3	1	6

Divide

5

2 4 3 1

8 9 6

Divide

3

2 1

4

6

8

9

Divide

1 element

1 2

Conquer

1 2

Conquer

1 2 3 4

6 8 9

Conquer

1 2 3 4 5 6 8 9

Quicksort Details

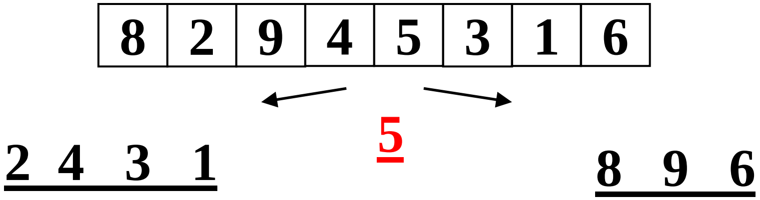
We have not explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But we want the two partitions to be about equal in size

- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - Median
  - Halve each time



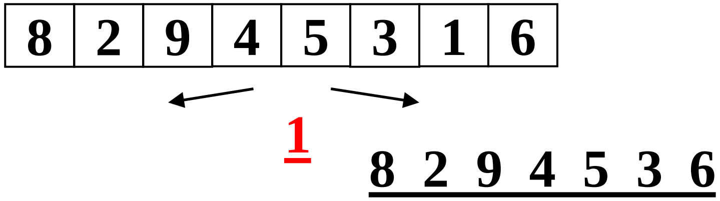
8 2 9 4 5 3 1 6

5

2 4 3 1

8 9 6

- Worst pivot?
  - Greatest/least element
  - Problem of size n - 1
  - O(n^2)



8 2 9 4 5 3 1 6

1

8 2 9 4 5 3 6

Quicksort: Potential Pivot Rules

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive):

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case occurs with approximately sorted input

- Pick random element in the range
  - Does as well as any technique
    - But random number generation can be slow
    - Still probably the most elegant approach

- Median of 3, (e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**)
  - Common heuristic that tends to work well

Partitioning

- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition in linear time in place

- One approach (there are slightly fancier ones):

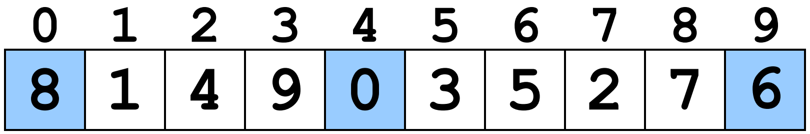
1. Swap pivot with `arr[lo]`
2. Use two fingers `i` and `j`, starting at `lo+1` and `hi-1`

```
3. while (i < j)
        if (arr[j] >= pivot) j--
        else if (arr[i] =< pivot) i++
        else swap arr[i] with arr[j]
```

4. Swap pivot with `arr[i]`

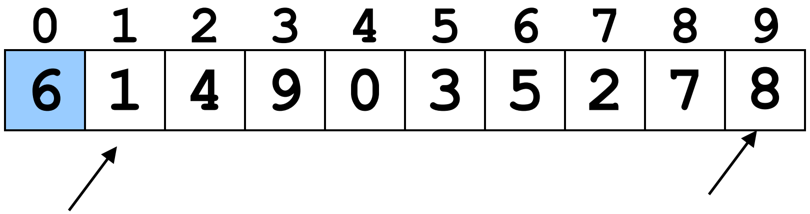
Quicksort Example

- Step One: Pick Pivot as Median of 3
  - `lo` = 0, `hi` = 10



0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

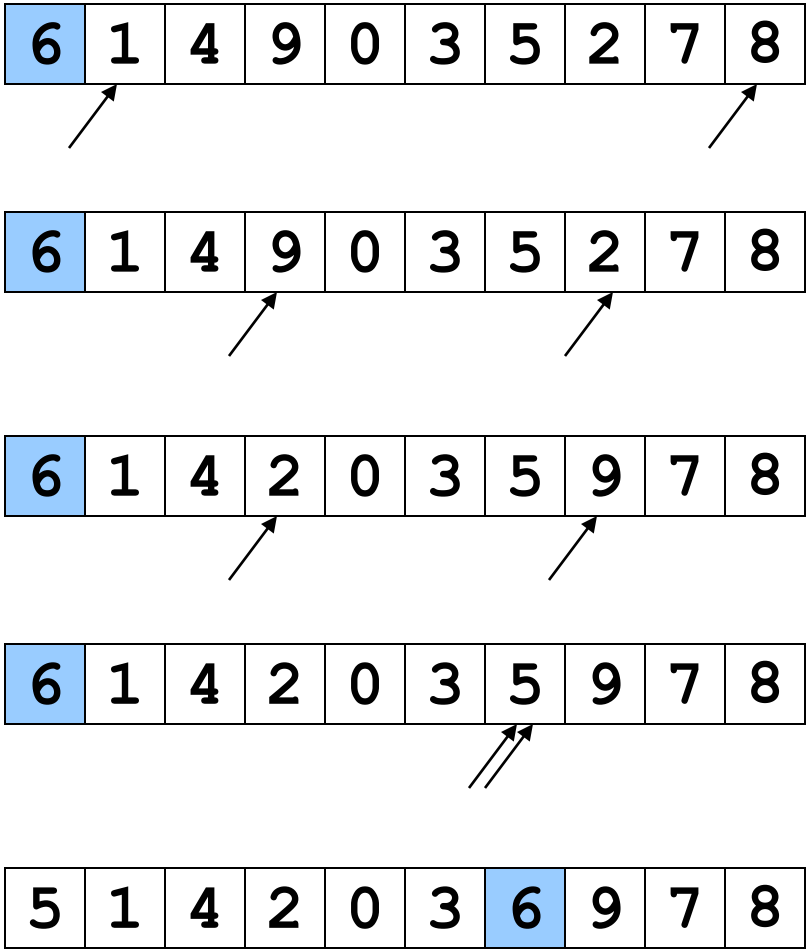
- Step Two: Move Pivot to the `lo` Position



0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Quicksort Example

Often have more than one swap during partition – this is a short example



Now partition in place	6	1	4	9	0	3	5	2	7	8
Move fingers	6	1	4	9	0	3	5	2	7	8
Swap	6	1	4	2	0	3	5	9	7	8
Move fingers	6	1	4	2	0	3	5	9	7	8
Move pivot	5	1	4	2	0	3	6	9	7	8

Quicksort Analysis

- Best-case: Pivot is always the median

  $T(0)=T(1)=1$

  $T(n)=2T(n/2) + n$ -- linear-time partition

  Same recurrence as mergesort: $O(n \log n)$

- Worst-case: Pivot is always smallest or largest element

  $T(0)=T(1)=1$

  $T(n) = 1T(n-1) + n$

  Basically same recurrence as selection sort: $O(n^2)$

- Average-case (e.g., with random pivot)
  - $O(n \log n)$ (see text)