CSE332: Data Abstractions
Lecture 8: Hashing

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Winter 2012
Conclusion of Balanced Trees

• Balanced trees make good dictionaries because they guarantee logarithmic-time \textbf{find}, \textbf{insert}, and \textbf{delete}
  – Essential and beautiful computer science
  – But only if you can maintain balance within the time bound

• \textbf{AVL trees} maintain balance by tracking height and allowing all children to differ in height by at most 1

• \textbf{B trees} maintain balance by keeping nodes at least half full and all leaves at same height

• Other great balanced trees (see text; worth knowing they exist)
  – \textbf{Red-black trees}: all leaves have depth within a factor of 2
  – \textbf{Splay trees}: self-adjusting; amortized guarantee; no extra space for height information
## Simple Implementations

For dictionary with \( n \) key/value pairs

<table>
<thead>
<tr>
<th>Method</th>
<th>insert</th>
<th>find</th>
<th>delete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unsorted linked-list</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>Unsorted array</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>Sorted linked list</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>Sorted array</td>
<td>( O(n) )</td>
<td>( O(\log n) )</td>
<td>( O(n) )</td>
</tr>
<tr>
<td>Balanced tree</td>
<td>( O(\log n) )</td>
<td>( O(\log n) )</td>
<td>( O(\log n) )</td>
</tr>
<tr>
<td>Magic array</td>
<td>( O(1) )</td>
<td>( O(1) )</td>
<td>( O(1) )</td>
</tr>
</tbody>
</table>

average case
Hash Tables

• Aim for constant-time **find**, **insert**, and **delete**
  – “On average” under some reasonable **assumptions**

• A hash table is an array of some fixed size

• Basic idea:

  key space (e.g., integers, strings)  

  hash function: \( \text{index} = h(\text{key}) \)  

  TableSize – 1  

  hash table  

  0  

  ...
**Hash Tables vs. Balanced Trees**

- In terms of a Dictionary ADT for just `insert`, `find`, `delete`, hash tables and balanced trees are just different data structures
  - Hash tables $O(1)$ on average (*assuming* few collisions)
  - Balanced trees $O(\log n)$ worst-case

- Constant-time is better, right?
  - Yes, but you need “hashing to behave” (must avoid collisions)
  - Yes, but `findMin`, `findMax`, `predecessor`, `successor` go from $O(\log n)$ to $O(n)$, `printSorted` from $O(n)$ to $O(n \log n)$

- **Moral**: If you need to frequently use operations based on sort order, then you may prefer a balanced BST instead.
Hash Tables

• There are $m$ possible keys ($m$ typically large, even infinite)
• We expect our table to have only $n$ items
• $n$ is much less than $m$ (often written $n << m$)

Many dictionaries have this property

– Compiler: All possible identifiers allowed by the language vs. those used in some file of one program

– Database: All possible student names vs. students enrolled

– AI: All possible chess-board configurations vs. those considered by the current player
Hash Functions

An ideal hash function:
- Is fast to compute
- “Rarely” hashes two “used” keys to the same index
  - Often impossible in theory; easy in practice
  - Will handle collisions in later

hash function:
index = h(key)

key space (e.g., integers, strings)

hash table

TableSize – 1
Who Hashes What

- Hash tables can be generic
  - To store elements of type `E`, we just need `E` to be:
    1. Comparable: order any two `E` (as with all dictionaries)
    2. Hashable: convert any `E` to an `int`

- When hash tables are a reusable library, the division of responsibility generally breaks down into two roles:

![Diagram showing the division of responsibility between client and hash table library]

- We will learn both roles, but most programmers “in the real world” spend more time as clients while understanding the library
More on Roles

Some ambiguity in terminology on which parts are “hashing”

Two roles must both contribute to minimizing collisions (heuristically)

• Client should aim for different ints for expected items
  – Avoid “wasting” any part of \( E \) or the 32 bits of the \( \text{int} \)
• Library should aim for putting “similar” \( \text{ints} \) in different indices
  – conversion to index is almost always “mod table-size”
  – using prime numbers for table-size is common
What to Hash?
We will focus on two most common things to hash: ints and strings

- If you have objects with several fields, it is usually best to hash most of the “identifying fields” to avoid collisions

- Example:

```java
class Person {
    String first; String middle; String last;
    Date birthdate;
}
```

- An inherent trade-off: hashing-time vs. collision-avoidance
Hashing Integers

- key space = integers

- Simple hash function:
  \[ h(key) = key \mod \text{TableSize} \]
  - Client: \( f(x) = x \)
  - Library \( g(x) = f(x) \mod \text{TableSize} \)
  - Fairly fast and natural

- Example:
  - TableSize = 10
  - Insert 7, 18, 41, 34, 10
  - (As usual, ignoring corresponding data)
Collision Avoidance

- With “x % TableSize” the number of collisions depends on
  - the ints inserted
  - TableSize

- Larger table-size tends to help, but not always
  - Example: 70, 24, 56, 43, 10
    with TableSize = 10 and TableSize = 60

- Technique: Pick table size to be prime. Why?
  - Real-life data tends to have a pattern,
  - “Multiples of 61” are probably less likely than “multiples of 60”
  - We will see some collision strategies do better with prime size
More Arguments for a Prime Size

If TableSize is 60 and...
- Lots of data items are multiples of 2, wasting 50% of table
- Lots of data items are multiples of 5, wasting 80% of table
- Lots of data items are multiples of 10, wasting 90% of table

If TableSize is 61...
- Collisions can still happen but 2, 4, 6, 8, … will fill table
- Collisions can still happen, but 5, 10, 15, 20, … will fill table
- Collisions can still happen but 10, 20, 30, 40, … will fill table

In general, if \( x \) and \( y \) are “co-prime” (means \( \gcd(x, y) == 1 \)),
then \( (a * x) \mod y == (b * x) \mod y \) if and only if \( a \mod y == b \mod y \)
- Good to have a TableSize that has
  no common factors with any “likely pattern” of \( x \)
What if key is not an int?

- If keys are not ints, the client must convert to an int
  - Trade-off: speed and distinct keys hashing to distinct ints

- Common and important example: Strings
  - Key space \( K = s_0 s_1 s_2 \ldots s_{m-1} \)
    - where \( s_i \) are chars: \( s_i \in [0,256] \)
  
  - Some choices: Which best avoid collisions?

1. \( h(K) = s_0 \mod \text{TableSize} \)

2. \( h(K) = \left( \sum_{i=0}^{m-1} s_i \right) \mod \text{TableSize} \)

3. \( h(K) = \left( \sum_{i=0}^{k-1} s_i \cdot 37^i \right) \mod \text{TableSize} \)
Combining Hash Functions

A few rules of thumb / tricks:

1. Use all 32 bits (careful, that includes negative numbers)

2. Use different overlapping bits for different parts of the hash
   – This is why a factor of $37^i$ works better than $256^i$
   – Example: “abcde” and “ebcda”

3. When smashing two hashes into one hash, use bitwise-xor
   – bitwise-and produces too many 0 bits
   – bitwise-or produces too many 1 bits

4. Rely on expertise of others; consult books and other resources

5. Advanced: If keys are known ahead of time, a perfect hash
Collision Resolution

Collision:
When two keys map to the same location in the hash table

We try to avoid it, but number-of-keys exceeds table size

So hash tables generally need to support collision resolution
Separate Chaining

Chaining:
All keys that map to the same
table location are kept in a list
(a.k.a. a “chain” or “bucket”)

As easy as it sounds

Example:
insert 10, 22, 107, 12, 42
with mod hashing
and TableSize = 10
**Separate Chaining**

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<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<th>7</th>
<th>8</th>
<th>9</th>
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Example:

insert 10, 22, 107, 12, 42 with mod hashing

and `TableSize = 10`
Thoughts on Separate Chaining

• Worst-case time for find?
  – Linear
  – But only with really bad luck or bad hash function
  – So not worth avoiding (e.g., with balanced trees at each bucket)
  • Keep small number of items in each bucket
  • Overhead of tree balancing not worthwhile for small n

• Beyond asymptotic complexity, some “data-structure engineering”
  – Linked list, array, or a hybrid
  – Move-to-front list (as in Project 2)
  – Leave one element in the table itself,
    to optimize constant factors for the common case
More Rigorous Separate Chaining Analysis

Definition: The load factor, $\lambda$, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}} \quad \leftarrow \text{number of elements}$$

Under chaining, the average number of elements per bucket is ___

So if some inserts are followed by random finds, then on average:

• Each unsuccessful $\text{find}$ compares against ____ items
• Each successful $\text{find}$ compares against _____ items

• How big should $\text{TableSize}$ be??
More Rigorous Separate Chaining Analysis

Definition: The load factor, $\lambda$, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}} \leftarrow \text{number of elements}$$

Under chaining, the average number of elements per bucket is $\lambda$

So if some inserts are followed by random finds, then on average:
- Each unsuccessful \texttt{find} compares against $\lambda$ items
- Each successful \texttt{find} compares against $\lambda/2$ items
- If $\lambda$ is low, find & insert likely to be $O(1)$
- We like to keep $\lambda$ around 1 for separate chaining
Separate Chaining Deletion?
Separate Chaining Deletion

- Not too bad
  - Find in table
  - Delete from bucket
- Delete 12
- Similar run-time as insert