CSE332: Data Abstractions

Lecture 3: Asymptotic Analysis

Tyler Robison (covering for James Forgarty)

Winter 2012

Overview

- Asymptotic analysis
  - Why we care
  - Big Oh notation
  - Examples
  - Caveats & miscellany
  - Evaluating an algorithm
  - Big Oh's family
  - Recurrence relations for analysis

What do we want to analyze?

- Correctness
- Performance: Algorithm’s speed or memory usage: our focus
  - Change in speed as the input grows
    - n increases by 1
    - n doubles
  - Comparison between 2 algorithms
- Security
- Reliability
- Sometimes other properties (‘stable’ sorts)

Gauging performance

- Uh, why not just run the program and time it?
  - Too much variability; not reliable:
    - Hardware: processor(s), memory, etc.
    - OS, version of Java, libraries, drivers
    - Choice of input
    - Programs running in the background, OS stuff, etc.: several executions on the same computer with the same settings may well yield different results
    - Implementation dependent
  - Timing doesn't really evaluate the algorithm; it evaluates its implementation in one very specific scenario
  - As computer scientists, we are more interested in the algorithm itself

Gauging performance (cont.)

- At the core of CS is a backbone of theory & mathematics
  - Examine the algorithm itself, mathematically, not the implementation
  - Reason about performance as a function of n; not just ‘it runs fast on this particular test file’
  - Be able to mathematically prove things about performance
- Yet, timing has its place
  - In the real world, we do want to know whether implementation A runs faster than implementation B on data set C
  - Ex: Benchmarking graphics cards
  - May do some timing in projects
- Evaluating an algorithm? Use asymptotic analysis
- Evaluating an implementation of hardware/software? Timing can be useful

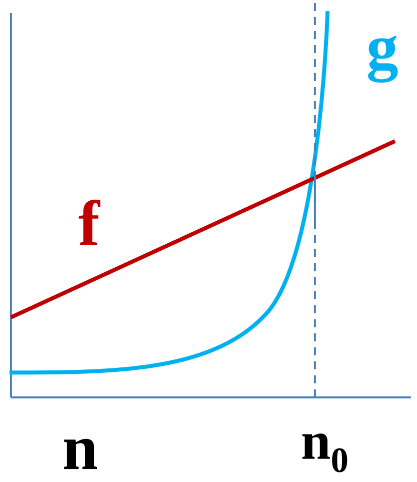
Overview

- Asymptotic analysis
  - ~~Why we care~~
  - Big Oh notation
  - Examples
  - Caveats & miscellany
  - Evaluating an algorithm
  - Big Oh's family
  - Recurrence relations for analysis

Big-Oh

- Say we're given 2 run-time functions f(n) & g(n) for input n
- The Definition: f(n) is in O(g(n)) iff there exist *positive* constants c and n_0 such that

  f(n) $\leq$ c g(n), for all $n \geq n_{0.}$



- The Idea: Can we find an n_0 such that g is always greater than f from there on out?

  c: We are allowed to multiply g by a constant value (say, 10) to make g larger (more on why this is here in a moment)

O(g(n)) is really a set of functions whose asymptotic behavior is less than or equal that of g(n)

Think of 'f(n) is in O(g(n))' as f(n) ≤ g(n) (sort of)

Big Oh (cont.)

- The Intuition:
  - Take functions f(n) & g(n), consider only the most significant term and remove constant multipliers:
    - $5n+3 \rightarrow n$
    - $7n+.5n^2+2000 \rightarrow n^2$
    - $300n+12+n\log n \rightarrow n\log n$
    - $-n \rightarrow$??? What does it mean to have a negative run-time?
  - Then compare the functions; if $f(n) \leq g(n)$, then $f(n)$ is in $O(g(n))$
  - Do NOT ignore constants that are not additions or multipliers:
    - n^3 is $O(n^2)$: FALSE
    - 3^n is $O(2^n)$: FALSE
  - When in doubt, refer to the definition (examples in a moment)

Examples

- True or false?

1. $4+3n$ is $O(n)$ — True
2. $n+2\log n$ is $O(\log n)$ — False
3. $\log n+2$ is $O(1)$ — False
4. n^{50} is $O(1.01^n)$ — True
5. There exists $\alpha>1.0$ s.t. — False

   α^n is $O(n^\beta)$

For some finite β

Examples (cont.)

- For $f(n)=4n$ & $g(n)=n^2$, prove $f(n)$ is in $O(g(n))$
  - A valid proof (for our purposes) is to find valid c & n_0
  - When n=4, f=16 & g=16; this is the crossing over point
  - Say $n_0 = 4$, and c=1
  - How many possible answers (c, n_0) are there?
    - *Infinitely many:

ex: $n_0 = 78$, and c=42

The Definition: f(*n*) is in O(g(*n*)) iff there exist *positive* constants *c* and n_0 such that

$$f(n) \le c\, g(n) \text{ for all } n \ge n_0.$$

Examples (cont.)

- For $f(n)=n^3$ & $g(n)=2^n$, prove f(n) is in O(g(n))
  - Possible answer: $n_0=11$, $c=1$

The Definition: f(*n*) is in O(g(*n*)) iff there exist *positive* constants *c* and n_0 such that

$$f(n) \le c\, g(n) \text{ for all } n \ge n_0.$$

What's with the c?

- To capture this notion of similar asymptotic behavior, we allow a constant multiplier (called c)
- Consider:

  $f(n)=7n+5$

  $g(n)=n$
- These have the same asymptotic behavior (linear), so f(n) is in O(g(n)) even though f is <u>always</u> larger
- There is no n_0 such that $f(n) \leq g(n)$ for all $n \geq n_0$
- The 'c' in the definition allows for that; it allows us to 'throw out constant factors'
- To prove f(n) is in O(g(n)), have $c=12$, $n_0=1$

Big Oh: Common Categories

From fastest to slowest

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic
$O(n)$	linear
$\mathrm{O}(\mathrm{n}\ \texttt{log}\ n)$	"n $\texttt{log}\ n$"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is an constant)
$O(k^n)$	exponential (where k is any constant > 1)

Usage note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some k>1"

– A savings account accrues interest exponentially (k=1.01?)

Where does $\log^2 n$ fit in?

Where does loglogn fit in?

Caveats

- Asymptotic complexity focuses on behavior of the algorithm for large *n* and is independent of any computer/coding trick, but results can be misleading
  - Example: $n^{1/10}$ vs. **`log`** n
    - Asymptotically $n^{1/10}$ grows more quickly
    - But the “cross-over” point is around $5 * 10^{17}$
    - So if you have input size less than 2^{58}, prefer $n^{1/10}$

More Caveats

- Even for more common functions, comparing O() for small n values can be misleading
  - Quicksort: O(nlogn) (expected)
  - Insertion Sort: $O(n^2)$(expected)
  - Yet in reality Insertion Sort is faster for small n's
  - We'll learn about these sorts later
- Usually talk about an algorithm being O(n) or whatever
  - But you can also prove bounds for entire problems
  - Ex: Sorting cannot take place faster than O(nlogn) in the worst case (assuming it's sequential and comparison-based; more on these later)

Miscellaneous

- Not uncommon to evaluate for:
  - Best-case
  - Worst-case
  - ‘Expected case’
- What are the run-times for BST lookup?
  - Best O(1) – find at root
  - Worst O(n) – tree is 1 long branch
  - ‘Expected’ O(logn) – complicated; see book

Notational Notes

- We say $(3n^2+17)$ **is in** $O(n^2)$
  - Confusingly, we also say/write:
    - $(3n^2+17)$ **is** $O(n^2)$
    - $(3n^2+17)$ = $O(n^2)$ (very common; in the book)
      - But it's not '=' as in 'equality':
      - We would never say $O(n^2) = (3n^2+17)$
- Perhaps the most accurate notation is

  $f(n) \in O(g(n))$

  - Because O(g(n)) is a set of functions

Analyzing code (worst case)

Basic operations take “some amount of” constant time:

- Arithmetic (fixed-width)
- Assignment to a variable
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a useful “lie”.)

Consecutive statements	Sum of times
Conditionals	Time of test plus slower branch
Loops	Sum of iterations
Calls	Time of call’s body
Recursion	Solve *recurrence equation*

Analyzing code

What are the run-times for the following code:

1. for(int i=0;i<n;i++) O(1) — O(n)
2. for(int i=0;i<=n+100;i+=14) O(1) — O(n)
3. for(int i=0;i<n;i++) for(int j=0;j<i;j++) O(1) — $O(n^2)$
4. for(int i=0;i<n;i++) for(int j=0;j<n;j++) O(n) — $O(n^3)$
5. for(int i=1;i<n;i*=2) O(1) — O(logn)
6. for(int i=0;i<n;i++) if(m(i)) O(n) else O(1) — Depends on m(); worst: $O(n^2)$

Big Oh's Family

- Big Oh: Upper bound: O(f(n)) is the set of all functions asymptotically less than or equal to f(n): '$\leq$' of functions
  - g(n) is in O(f(n)) if there exist constants c and n_0 such that

    $g(n) \leq c\, f(n)$ for all $n \geq n_0$

- Big Omega: Lower bound: Ω(f(n)) is the set of all functions asymptotically greater than or equal to f(n): '$\geq$' of functions
  - g(n) is in Ω(f(n)) if there exist constants c and n_0 such that

    $g(n) \geq c\, f(n)$ for all $n \geq n_0$

- Big Theta: Tight bound: θ(f(n)) is the set of all functions asymptotically equal to f(n): '=' of functions
  - Intersection of O(f(n)) and Ω(f(n)) (use *different constants*)

Regarding use of terms

Common error is to say $O(\mathrm{f}(n))$ when you mean $\theta(\mathrm{f}(n))$

- People often say O() to mean a tight bound
- Say we have f(n)=n; we could say f(n) is in O(n), which is true, but only conveys the upper-bound
- Somewhat incomplete; instead say it is $\theta(n)$
- This gives us a tighter bound

Less common notation:

- "little-oh": like "big-Oh" but strictly less than
  - Example: n is $o(n^2)$ but not $o(n)$
- "little-omega": like "big-Omega" but strictly greater than
  - Example: n is $\omega(\texttt{log}\ n)$ but not $\omega(n)$

Recurrence Relations

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
  - Conceptually, in each recursive call we:
    - Perform some amount of work, call it $w(n)$
    - Call the function recursively with a smaller portion of the list

So, if we do $w(n)$ work per step, and reduce the n in the next recursive call by 1, we do total work:

$$T(n)=w(n)+T(n-1)$$

With some base case, like $T(1)=5=O(1)$

Recursive version of sum array

Recursive:

- Recurrence is $k + k + \ldots + k$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

Recurrence Relation: $T(n) = O(1) + T(n\text{-}1)$

Recurrence Relations (cont.)

Say we have the following recurrence relation:

$T(n)=2+T(n-1)$

$T(1)=5$

Now we just need to solve it; that is, reduce it to a closed form

Start by writing it out:

$T(n)=2+T(n-1)=2+2+T(n-2)=2+2+2+T(n-3)$

$=2+2+2+\ldots+2+T(1)=2+2+2+\ldots+2+5$

$=2k+5$, where k is the # of times we expanded T()

We expanded it out n-1 times, so

$T(n)=2(n-1)+5=2n+3=O(n)$

Example: Find k

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   ???
}
```

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6ish steps = $O(1)$

Worst case: 6ish*(arr.length)

= O(arr.length) = O(n)

Binary search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

- Can also be done non-recursively (same run-time)

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)          return false;
   if(arr[mid]==k)     return true;
   if(arr[mid]< k)     return help(arr,k,mid+1,hi);
   else                return help(arr,k,lo,mid);
}
```

Binary search

Best case: 8ish steps = O(1)

Worst case:

T(n) = 10ish + T(n/2) where n is hi-lo

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

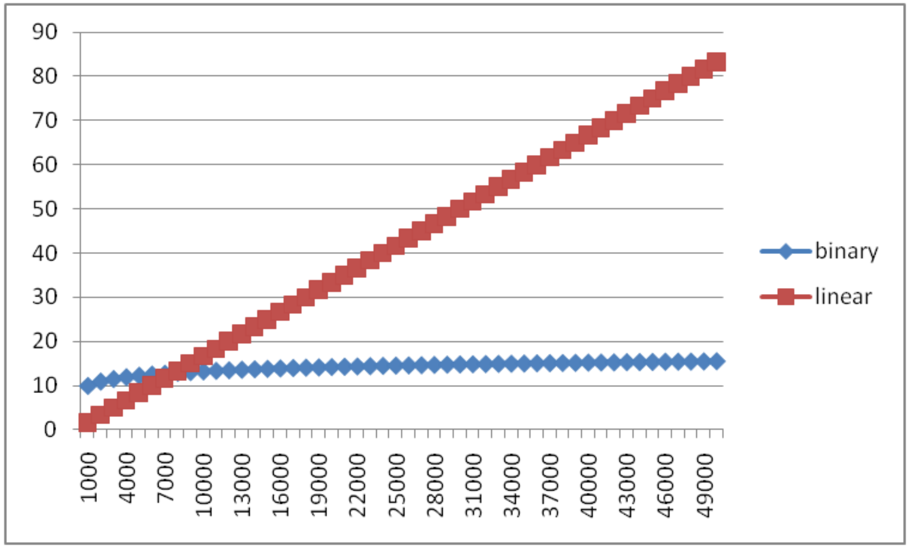
Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   – $T(n) = 10 + T(n/2)$ $T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   – $T(n) = 10 + 10 + T(n/4)$
   $= 10 + 10 + 10 + T(n/8)$
   $= \ldots$
   $= 10k + T(n/(2^k))$ where k is the # of expansions
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   – $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   – So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   – So $T(n)$ is $O(\log n)$

Linear vs Binary Search

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - Given the constants, linear search could still be faster for small values of n

Example w/ hypothetical constants:



What about a binary version of sum?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2) = O(n)$

- (Proof left as an exercise)

"Obvious": have to read the whole array

- You can't do better than *O(n)*
- Or can you…
  - We'll see a parallel version of this much later
  - With ∞ processors, $T(n) = O(1) + 1T(n/2) = O(\log n)$

Another example

- $T(n)=n + 2T(n/2)$, $T(1)=c$
  - Any guesses as to what algorithm(s) this represents?
    - Mergesort & quicksort (assuming good pivot selection)
  - Any guesses as to what the closed form for this is?
    - $O(n\log n)$

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic
$T(n) = O(n) + T(n/2)$	linear
$T(n) = O(n) + 2T(n/2)$	$O(\text{n } \mathbf{\texttt{log}} \text{ n})$

Note big-Oh can also use more than one variable (graphs: vertices & edges)

- Example: you can (and will in proj3!) sum all elements of an *n*-by-*m* matrix in $O(nm)$