CSE332: Data Abstractions

Lecture 20: Parallel Prefix, Pack, and Sorting

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Outline

Done:
- Simple ways to use parallelism for counting, summing, finding
- Analysis of running time and implications of Amdahl's Law

Now: Clever ways to parallelize more than is intuitively possible
- Parallel prefix:
  - This "key trick" typically underlies surprising parallelization
  - Enables other things like packs
- Parallel sorting: quicksort (not in place) and mergesort
  - Easy to get a little parallelism
  - With cleverness can get a lot

The prefix-sum problem

Given `int[] input`, produce `int[] output` where `output[i]` is the sum of `input[0]+input[1]+…+input[i]`

Sequential can be a CSE142 exam problem:

```
int[] prefix_sum(int[] input){
  int[] output = new int[input.length];
  output[0] = input[0];
  for(int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Does not seem parallelizable
- Work: $O(n)$, Span: $O(n)$
- This *algorithm* is sequential, but a *different algorithm* has Work: $O(n)$, Span: $O(\log n)$

Parallel prefix-sum

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\log n)$ span
  - So in total there is $O(n)$ work and $O(\log n)$ span
  - So like with array summing, parallelism is $n/\log n$
    - An exponential speedup
- First pass builds a tree bottom-up: the "up" pass
- Second pass traverses the tree top-down: the "down" pass

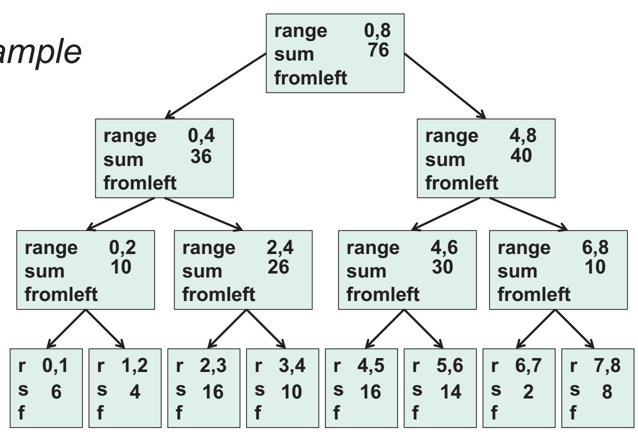
Local bragging

Historical note:
- Original algorithm due to R. Ladner and M. Fischer at UW in 1977
- Richard Ladner joined the UW faculty in 1971 and hasn't left


1968? 1973?

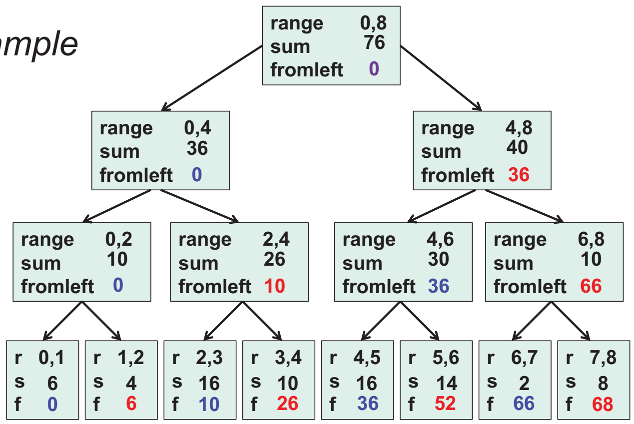

recent

Example



`input`	6	4	16	10	16	14	2	8
`output`								

Example



`input`	6	4	16	10	16	14	2	8
`output`	6	10	26	36	52	66	68	76

The algorithm, part 1

1. Up: Build a binary tree where
   – Root has sum of the range [**`x,y`**)
   – If a node has sum of [**`lo,hi`**) and **`hi>lo`**,
     • Left child has sum of [**`lo,middle`**)
     • Right child has sum of [**`middle,hi`**)
     • A leaf has sum of [**`i,i+1`**), i.e., **`input[i]`**

This is an easy fork-join computation: combine results by actually building a binary tree with all the range-sums
– Tree built bottom-up in parallel
– Could be more clever with an array like with heaps

Analysis: $O(n)$ work, $O(\log n)$ span

The algorithm, part 2

2. Down: Pass down a value **`fromLeft`**
   – Root given a **`fromLeft`** of **`0`**
   – Node takes its **`fromLeft`** value and
     • Passes its left child the same **`fromLeft`**
     • Passes its right child its **`fromLeft`** plus its left child's **`sum`** (as stored in part 1)
   – At the leaf for array position **`i`**, **`output[i]=fromLeft+input[i]`**

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result
– Leaves assign to **`output`**
– Invariant: **`fromLeft`** is sum of elements left of the node's range

Analysis: $O(n)$ work, $O(\log n)$ span

Sequential cut-off

Adding a sequential cut-off is easy as always:

• Up:
  just a sum, have leaf node hold the sum of a range

• Down:

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```


Parallel prefix, generalized

Just as sum-array was the simplest example of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

• Minimum, maximum of all elements to the left of **`i`**

• Is there an element to the left of **`i`** satisfying some property?

• Count of elements to the left of **`i`** satisfying some property
  – This last one is perfect for an efficient parallel pack…
  – Perfect for building on top of the "parallel prefix trick"

• We did an *inclusive* sum, but *exclusive* is just as easy

Pack

[Non-standard terminology]

Given an array **`input`**, produce an array **`output`** containing only elements such that **`f(elt)`** is **`true`**

Example: **`input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`**
`f: is elt > 10`
`output [17, 11, 13, 19, 24]`

Parallelizable?
– Finding elements for the output is easy
– But getting them in the right place seems hard

Parallel prefix to the rescue

1. Parallel map to compute a bit-vector for true elements

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. Parallel-prefix sum on the bit-vector

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. Parallel map to produce the output

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){
  if(bits[i]==1)
    output[bitsum[i]-1] = input[i];
}
```


Pack comments

- First two steps can be combined into one pass
  - Just using a different base case for the prefix sum
  - No effect on asymptotic complexity
- Can also combine third step into the down pass of the prefix sum
  - Again no effect on asymptotic complexity
- Analysis: $O(n)$ work, $O(\texttt{log}\ n)$ span
  - 2 or 3 passes, but 3 is a constant
- Parallelized packs will help us parallelize quicksort…

Quicksort review

Recall quicksort was sequential, in-place, expected time $O(n\ \texttt{log}\ n)$

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

How should we parallelize this?

Quicksort

	Best / expected case *work*
1. Pick a pivot element	**O(1)**
2. Partition all the data into:	**O(n)**
A. The elements less than the pivot	
B. The pivot	
C. The elements greater than the pivot	
3. Recursively sort A and C	**2T(n/2)**

Easy: Do the two recursive calls in parallel

- Work: unchanged of course $O(n\ \texttt{log}\ n)$
- Span: now $T(n) = O(n) + 1T(n/2) = O(n)$
- So parallelism (i.e., work / span) is $O(\texttt{log}\ n)$

Doing better

- $O(\texttt{log}\ n)$ speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort 10^9 elements 30 times faster
- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong ☺
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law
- Already have everything we need to parallelize the partition…

Parallel partition (not in place)

Partition all the data into:

A. The elements less than the pivot
B. The pivot
C. The elements greater than the pivot

- This is just two packs!
  - We know a pack is $O(n)$ work, $O(\texttt{log}\ n)$ span
  - Pack elements less than pivot into left side of **`aux`** array
  - Pack elements greater than pivot into right size of **`aux`** array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity
- With $O(\texttt{log}\ n)$ span for partition, the total span for quicksort is
  $T(n) = O(\texttt{log}\ n) + 1T(n/2) = O(\texttt{log}^2\ n)$

Example

- Step 1: pick pivot as median of three

8	1	4	9	0	3	5	2	7	6

- Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  - Fancy parallel prefix to pull this off not shown

1	4	0	3	5	2				

1	4	0	3	5	2	6	8	9	7

- Step 3: Two recursive sorts in parallel
  - Can sort back into original array (like in mergesort)

Now mergesort

Recall mergesort: sequential, not-in-place, worst-case $O(n \log n)$

1. **Sort left half and right half** **2T(n/2)**
2. **Merge results** **O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the span to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, parallelism is $O(\log n)$
- To do better, need to parallelize the merge
  - The trick won't use parallel prefix this time

Parallelizing the merge

Need to merge two *sorted* subarrays (may not have the same size)

0	1	4	8	9

2	3	5	6	7

Idea: Suppose the larger subarray has n elements. In parallel:

- Merge the first $n/2$ elements of the larger half with the "appropriate" elements of the smaller half
- Merge the second $n/2$ elements of the larger half with the rest of the smaller half

Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

1. Get median of bigger half: $O(1)$ to compute middle index

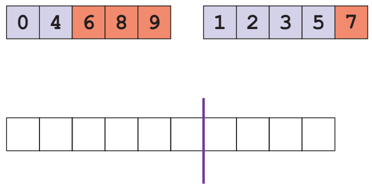
Parallelizing the merge

0	4	6	8	9

1	2	3	5	7

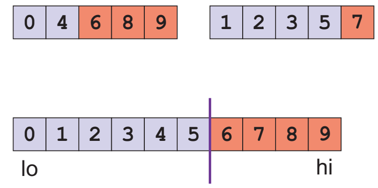
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: $O(\log n)$ to do binary search on the sorted small half

Parallelizing the merge



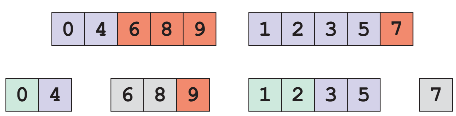
1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: $O(\log n)$ to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: $O(1)$

Parallelizing the merge



1. Get median of bigger half: $O(1)$ to compute middle index
2. Find how to split the smaller half at the same value as the left-half split: $O(\log n)$ to do binary search on the sorted small half
3. Size of two sub-merges conceptually splits output array: $O(1)$
4. Do two submerges in parallel

The Recursion



When we do each merge in parallel, we split the bigger one in half and use binary search to split the smaller one

Analysis

- Sequential recurrence for mergesort:
  $T(n) = 2T(n/2) + O(n)$ which is $O(n \log n)$
- Doing the two recursive calls in parallel but a sequential merge:
  Work: same as sequential Span: $T(n)=1T(n/2)+O(n)$ which is $O(n)$
- Parallel merge makes work and span harder to compute
  - Each merge step does an extra $O(\log n)$ binary search to find how to split the smaller subarray
  - To merge n elements total, do two smaller merges of possibly different sizes
  - But worst-case split is $(1/4)n$ and $(3/4)n$
    - When subarrays same size and "smaller" splits "all" / "none"

Analysis continued

For just a parallel merge of n elements:
- Work is $T(n) = T(3n/4) + T(n/4) + O(\log n)$ which is $O(n)$
- Span is $T(n) = T(3n/4) + O(\log n)$, which is $O(\log^2 n)$
- (neither bound is immediately obvious, but "trust me")

So for mergesort with parallel merge overall:
- Work is $T(n) = 2T(n/2) + O(n)$, which is $O(n \log n)$
- Span is $T(n) = 1T(n/2) + O(\log^2 n)$, which is $O(\log^3 n)$

So parallelism (work / span) is $O(n / \log^2 n)$
- Not quite as good as quicksort's $O(n / \log n)$
  - But worst-case guarantee
- And as always this is just the asymptotic result