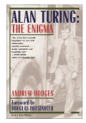
Your Chance to Win a Turing Award

- It is generally believed that P ≠ NP, *i.e.* there are problems in NP that are not in P
  - But no one has been able to show even one such problem!
  - This is the fundamental open problem in theoretical computer science
  - Nearly everyone has given up trying to prove it. Instead, theoreticians prove theorems about what follows once we assume P ≠ NP !

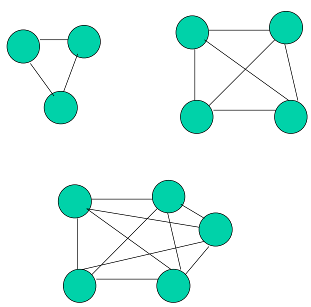


Alan Turing
(1912-1954)

Outline: Day 2

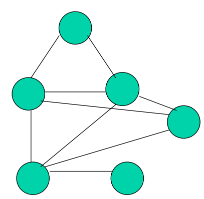
- We've seen that there are a bunch of problems that seem to be hard.
- Today we'll see how these problems relate to one another.
- Def: P_1 is reducible to P_2 if there is a conversion from an instance X of P_1 to an instance Y of P_2 such that P_1 is yes for X iff P_2 is yes for Y.

Clique



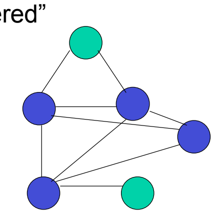
k-Clique problem

- Is there a clique of size *k* in the graph?



Vertex cover

- Set of vertices so that each edge is "covered"



Vertex cover of size 4

How are they related?

- How might we reduce clique to vertex cover?
- That is, given a clique problem (G, k), how can we turn it into a vertex cover problem?
- *Once we do this reduction, we know we can always solve vertex cover given solution to clique!*

Clique to Vertex Cover

- We can reduce Clique to Vertex Cover.
- Given an input (G, k) to Clique:
  - Build graph G complement
  - Let k' = n – k
- Vertex Cover is "as hard as" Clique.

→

- If G has a k Clique then G' has a k' cover:
  - Let C be the clique of size k. Let the cover be V-C. Then clearly every edge outside C is covered, and in G' there are no edges in C.
  - Size is n-k

←

- If G' has a k' cover then G has a k Clique:
  - Let D be a cover in G' of size k'. Then there are no edges in V-D, since otherwise they wouldn't be covered. Therefore, V-D is a clique in G.
  - Size of clique is n-k'.

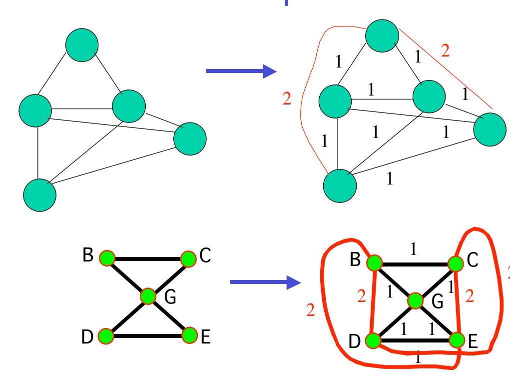
TSP

- Travelling Salesman Problem:
  - Given complete weighted graph G, integer k.
  - Is there a cycle that visits all vertices with cost <= k?
- One of the canonical problems.
- Note difference from Hamiltonian cycle: graph is complete, and we care about weight.

Hamiltonian Cycle to TSP

- We can reduce Hamiltonian Cycle to TSP.
- Given graph G=(V, E):
  - Construct complete graph G' on N vertices with edge weights: 1 if (u, v) in E, 2 otherwise.
  - Let k = N.
- TSP is "as hard as" Hamiltonian cycle.

Example



Proof

- If G has a Hamiltonian Cycle then G' has a tour of weight N.
  - The cycle is the tour, since there are N edges of weight 1
- If G' has a tour of weight N, then G has a Hamiltonian Cycle.
  - The tour is the cycle, since it must contain N edges, each edge must weigh 1, and thus must have been in original graph

Ham. Cycle to Longest Path

- Recall, Longest Path: Given directed graph G, start node s, and integer k. Is there a simple path from s of length >= k?
- We'll use Directed Hamiltonian Cycle.

The reduction

- Given a directed graph G, want to find Ham. Cycle
- Convert to Longest path
  - Pick any node as start vertex s.
  - Create a new node t. For every edge (u, s), add an edge (u, t). Let k = N.
- Longest Path is "as hard as" Ham. Cycle

Proof

- If G has a Ham. Cycle, then G' has a path of length k from s.
  - Follow the cycle starting at s, at the last step go to t instead of s.
- If G' has a path of length k from s, then G has a Ham. Cycle.
  - Path must have hit every node exactly once, and last step in path could have formed cycle in G.

NP-completeness

- We've seen that there are seemingly hard problems. That's kind of interesting.
- The really interesting part: A large class of these are equivalent. Solving one would give a solution for all of them!

More on NP-completeness

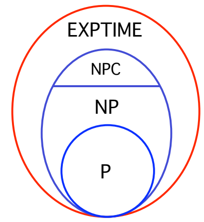
- The pairs I picked weren't important. There is a large class of problems, called NP-complete, such that *any* one can be reduced to *any* other.
- So given an algorithm for any NP-complete problem, all the others can be solved.
- Conversely, if we can prove there is no efficient algorithm for one, then there are no efficient algorithms for any.

NP-Complete Problems

- The “hardest” problems in NP are called **NP-complete**
  - If any NP-complete problem is in P, then all of NP is in P
- Examples:
  - Hamiltonian circuit
  - Traveling salesman: find the shortest path that visits all nodes in a weighted graph (okay to repeat edges & nodes)
  - Graph coloring: can the vertices of a graph be colored using K colors, such that no two adjacent vertices have the same color?
  - Crossword puzzle construction: can a given set of 2N words, each of length N, be arranged in an NxN crossword puzzle?

P, NP, and Exponential Time Problems

- All **currently known** algorithms for NP-complete problems run in **exponential** worst case time
  - Finding a polynomial time algorithm for any NPC problem would mean:
- Diagram depicts relationship between P, NP, and EXPTIME (class of problems that **provably require** exponential time to solve)



It is believed that
$P \neq NP \neq EXPTIME$

Coping with NP-Completeness

1. Settle for algorithms that are fast on average: Worst case still takes exponential time, but doesn’t occur very often.
   But some NP-Complete problems are also average-time NP-Complete!
2. Settle for fast algorithms that give near-optimal solutions: In traveling salesman, may not give the cheapest tour, but maybe good enough.
   But finding even approximate solutions to some NP-Complete problems is NP-Complete!
3. Just get the exponent as low as possible! Much work on exponential algorithms for satisfiability: in practice can often solve circuits with 1,000+ inputs
   But even $2^{n/100}$ will eventual hit the exponential curve!