# CSE 326: Proving asymptotic comparisons

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## 1 A little-o definition

### 1.1 First, Big-$\Omega$

$f(n) \in \Omega(g(n))$ means that there is at least one $c > 0$ and some $n_0$ such for all $n > n_0$, $f(n) \geq c \cdot g(n)$.

### 1.2 little-o

$f(n) \in o(g(n))$ means that for all $c > 0$ there exists some $n_0$ such that for all $n > n_0$, $f(n) < c \cdot g(n)$.

### 1.3 BTW, another way of looking at little-o

little-o means that there's no $c$ that will satisfy the Big-$\Omega$ condition, since *all* $c$ don't.

### 1.4 Alternate little-o definition (Warning: Calculus)

For any $c$, and for sufficiently large $n$, $f(n) < c \cdot g(n)$.

In other words,

$$\begin{aligned} & f(n) \in o(g(n)) \\ \iff \ & \lim_{n\to\infty} \tfrac{f(n)}{g(n)} = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} & f(n) \in \omega(g(n)) \\ \iff \ & \lim_{n\to\infty} \tfrac{f(n)}{g(n)} = \infty \end{aligned}$$

And, interestingly,

$$\begin{aligned} & f(n) \in \Theta(g(n)) \\ \iff \ & \lim_{n\to\infty} \frac{f(n)}{g(n)} = k \end{aligned}$$

for some *finite* constant $k > 0$.

(see p. 43 of the Weiss book)

## 2 First proof: $n \in o(n^2)$.

We use the limit of the fraction format:

$$\begin{aligned}
& \lim_{n\to\infty} \frac{n}{n^2} \\
= & \lim_{n\to\infty} \frac{1}{n} \quad \text{cancel out } n \\
= & \ 0
\end{aligned}$$

So it's true.

## 3 Generalization - useful theorem

Let $g(n)$ be a monotonically increasing function (mainly $\lim_{n\to\infty} g(n) = \infty$).

Then $cdotg(n) \in o(f(n) \cdot g(n)$ if $\lim_{n\to\infty} g(n) = \infty$.

Proof: use the fraction format.

$$\begin{aligned}
& \lim_{n\to\infty} \frac{g(n)}{f(n) \cdot g(n)} \\
= & \lim_{n\to\infty} \frac{1}{f(n)} \quad \text{cancel out } g(n) \\
= & \ 0 \quad \text{if } f(n) \text{ goes to } \infty
\end{aligned}$$

## 4 An easy one now: $n^k \in o(n^{k+\varepsilon})$ if $k, \varepsilon > 0$

$n^{k+\varepsilon} = n^k n^\varepsilon$.

Clearly, $\lim_{n\to\infty} n^\varepsilon = \infty$ if $\varepsilon > 0$. So, by our above theorem, $n^k \in o(n^\varepsilon n^k)$.

## 5 Same idea: $\log^k n \in o(\log^{k+\varepsilon} n)$ if $k, \varepsilon > 0$

$\log^{k+\varepsilon} n = (\log^k n)(\log^\varepsilon n)$. And we have the same idea as above.

## 6 Another easy one: $n \in o(n \cdot \log n)$

Follows from our theorem, and the fact that $\lim_{n\to\infty} \log n = \infty$.

## 7 l'Hôpital's Rule

$$\begin{aligned}
& \lim_{n\to\infty} \frac{f(n)}{g(n)} \\
\iff & \lim_{n\to\infty} \frac{f'(n)}{g'(n)}
\end{aligned}$$

$f'(n)$ is the first derivative of $f(n)$.

## 8 Using l'Hôpital's Rule to show $\log n \in o(n)$

Using the fraction format,

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\log n}{n} \\
= & \lim_{n\to\infty} \frac{1/n}{1} \quad \text{by l'Hôpital's Rule} \\
= & \lim_{n\to\infty} \frac{1}{n} \\
= & \ 0
\end{aligned}$$

## 9 $\log^i \in o(n^j)$ for $i, j > 0$

Look at what happens with the fractional format:

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\log^i n}{n^j} \\
= & \lim_{n\to\infty} \frac{(\log^{i-1} n)(1/n)i}{jn^{j-1}} \quad \text{the Chain rule} \\
= & \lim_{n\to\infty} \frac{\log^{i-1} n)i}{jn^j} \\
= & \lim_{n\to\infty} \frac{\log^{i-1} n)}{n^j} \quad i/j \text{ is just a constant}
\end{aligned}$$

Note: taking $i/j$ out is kind of sloppy, since we used an $=$ sign. But, it is valid given that we're only concerned about 0, some finite constant or $\infty$.

Now, we can prove it inductively (we just proved the induction step). The base case is just $\log^i \in o(n^j)$ for $j > 0$ and $0 < i \leq 1$, which is pretty straightforward from what we've already done.

# 10 Similar idea: $n^i \in o(j^n)$ for $i, j > 0$

The induction step:

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{n^i}{j^n} \\
= & \lim_{n\to\infty} \frac{n^i}{e^{n \ln j}} \quad \text{by log rules} \\
= & \lim_{n\to\infty} \frac{in^{i-1}}{(\ln j)e^{n \ln j}} \\
= & \lim_{n\to\infty} \frac{in^{i-1}}{(\ln j)e^{n \ln j}} \\
= & \lim_{n\to\infty} \frac{n^{i-1}}{e^{n \ln j}} \quad \text{eliminate constants} \\
= & \lim_{n\to\infty} \frac{n^{i-1}}{j^e}
\end{aligned}
$$

## 10.1 A funky theorem: $\log f(n) \in o(\log g(n)) \implies f(n) \in o(g(n)^c)$ for any $c > 0$

Proof: By the definition of little-o,

$$
\begin{aligned}
& \log f(n) \in o(\log g(n)) \\
\rightarrow & \forall c > 0 \exists n_0 \forall n > n_0 . \log f(n) < c \cdot \log g(n)
\end{aligned}
$$

Now, we add a constant to the right side of the inequality, which preserves the little-o relation. We obtain

$$\forall k \forall c > 0 \exists n_0 \forall n > n_0 . \log f(n) < c \cdot \log g(n) + k$$

So, exponentiating both sides of the inequality.

$$
\begin{aligned}
& \forall k \forall c > 0 \exists n_0 \forall n > n_0 . \log f(n) < c \cdot \log g(n) + k \\
\rightarrow & \forall k \forall c > 0 \exists n_0 \forall n > n_0 . f(n) < 2^k g(n)^c \\
\rightarrow & f(n) \in o(g(n)^c)
\end{aligned}
$$

We reached the definition for little-o, since $2^k$ can take on all positive values for some $k$

A corrolary is $\log f(n) \in o(\log g(n)) \implies f(n) \in o(g(n))$, by simply selecting $c = 1$.

Note that the reverse is not necessarily true. i.e. if $f(n) \in o(g(n))$ we don't necessarily know that $\log f(n) \in o(\log g(n))$. Can you think of a counterexample?

## 11 Interlude: $\log \log n \in o(\log n)$.

We do this using the fractional format, and by substituting $m = 2^n$:

$$\begin{aligned} & \lim_{n\to\infty} \frac{\log\log n}{\log n} \\ = & \lim_{n\to\infty} \frac{\log m}{m} \quad \text{do the substitution} \end{aligned}$$

and we know $\log m \in o(m)$ from before.

Note that it's important that $\lim_{n\to\infty} \log n = \infty$, otherwise the substitution wouldn't necessarily be valid.

## 12 Using funky theorem: $n^k \in o((\log n)^{\log n})$ for any $k$

We take log of both sides. Now, it turns out that

$$k \log n \in o((\log \log n)(\log n))$$

because the left side is $\Theta(\log n)$, while the right side has an extra $(\log \log n)$ factor on it.

And fortunately we have

$$\begin{aligned} k \log n &= \log(n^k) \\ \text{and } (\log\log n)(\log n) &= \log((\log n)^{\log n}) \end{aligned}$$

So, we can expontentiate both sides using the funky theorem, and get our answer.

## 13 Using funky theorem: $(\log n)^{\log n} \in o(2^k)$

Same idea as the previous one.