Pumping Lemma

A regular language $L$

$\exists p \quad \forall w \in L \quad |w| \geq p \Rightarrow$

$\exists x, y, z \in \Sigma^* \quad \text{s.t.}$

$w = xyz$

$y \neq \epsilon$

$|xy| \leq p$

$\forall i > 0 \quad xy^i z \in L$
Proof:

Let $L$ be regular, then there exists a DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepting $L$. Let $p \geq |Q|$. Let $w$ be any $w \in L$. If $|w| \leq p$, vacuous. If $|w| > p$, let $q_i$ be the state reached after reading $i$ letters, $1 \leq i \leq p$. Let $q_i$ be the state reached after reading $i$ letters of $w$. Then $p+1$ states for that case.
By Pigeon hole principle, \( \exists i < j \) such that \( x_i = x_j \). Let \( x = 1 \leq i \) letter of \( u \), \( y = (i+1) \) th through \( j \) th letter of \( u \), \( z = z \geq j \). 

\[
\begin{align*}
M & \text{ accept } x \\
M & \text{ accept } x \ y \\
M & \text{ accept } x \ y \ z \\
\vdots \\
M & \text{ accept } x \ y \ k = 6 \ k = 0
\end{align*}
\]
\[ l = \sum_{a=1}^{30} a^2 \]

Key Idea: perfect squares become increasingly sparse, but PL => at most p gap between members
\[ L = \{ a^{n^2} \mid n > 0 \} \subseteq \mathbb{A} \]

Suppose \( L \) is regular. By P.L.

\[ \exists p \quad \text{let } w = a^{p^2} \text{ by P.L.} \]

\[ \exists xyz \text{ s.t. } w = xye \]

\[ 0 < |y| \leq p \]

\[ xy^2z = a^{p^2} + |y| \]

\[ (p+1)^2 = p^2 + 2p + 1 \]

\[ p^2 + |y| \leq p^2 + p < p^2 + 2p + 1 \]

\[ \therefore xy^2z \notin L \]
\[ L = \{ a^n b^n | n \geq 0 \} \]

if \( L \) is regular then by P.L.
exists \( \exists p \) such ...
\[
W = a^p b^p
\]

\[ \exists x, y, z \leq 3^p \]

Let \( x + y + z = w \)

\[ |x| \geq 0 \]

\[ |x + y| \leq p \]

\[ x = -a_i \text{ for some } 0 \leq i < p \]

\[ y = a_j \text{ for some } 1 \leq j \leq p \]

\[ z = a^p - i - j \]

\[ xy^2z = a^p + j \]

\[ \therefore L \text{ is not regular} \]
\[ L = \{ w \mid \#a(w) = \#6(w) \} \]

\[ L \cap a^* b^* = \{ a^n b^n \mid n \geq 3 \]  
\text{regular}

\[ \text{not regular} \]

\[ \therefore \text{By closure of regular languages under intersection, } L \text{ cannot be regular.} \]