Regular expressions over $\Sigma$

$\emptyset$ is an r.e.
$\varepsilon$ .... r.e.

$\alpha$ .... for each $\alpha \in \Sigma$

if $R_1$ & $R_2$ are r.e.,
then so are

$(R_1 \cup R_2)$
$(R_1 \cdot R_2)$
$(R_1^*)$

The language denoted by $R$, $L(R)$
is:

$L(\emptyset) = \emptyset$
$L(\varepsilon) = \varepsilon \cdot \varepsilon$

$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$
Theorem:
A regular expression $R \exists$ an NFA $\text{Mr}$ s.t. $L(R) = L(\text{CMR})$

Proof:
By induction on $k$, the # of $\cup, \cdot, \ast$ operators in $R$

Base cases ($k=0$):
Then $R$ is "$\phi"$, "$\epsilon$", or "$a" for $a \in \Sigma$.

Explicitly give simple NFA's recognizing $\phi$, \{E\}, and \{a\} for each $a \in \Sigma$ (details omitted)

Induction Step ($R$ has $k > 0$ operators)

I.H.: assume that for all regular expressions $R'$ with $\leq k-1$ operators,
$\exists$ NFA $\text{Mr}'$ recognizing $L(R')$

$R$ has $k > 0$ operators. So $R \in (R_1, U R_2)$ or $(R_1 \cdot R_2)$ or $(R_1)^*$
where $R_1, (PR_2$ if any) have $\leq k-1$ operators. By I.H., $\exists Mr_1, (CMR_2) s.t.
L(R_i) = L(\text{CMR}_i), i=1,2$. Modify/join it/them as in previous proofs of closure under $\cup, \cdot, \ast$ to get $\text{Mr}$ s.t. $L(R) = L(\text{Mr})$. 
Example

$$(ab)^* u a$$
Converse?

For every D/NFA $\exists$ reg expr defining the same language

\[ \text{Diagram 1} \rightarrow (ab)^* \]

\[ \text{Diagram 2} \]
divisible by 5

pattern?

\[
\begin{align*}
0 & \rightarrow 1 & \rightarrow 2 & \rightarrow 4 & \rightarrow 3 & \rightarrow 1 & \rightarrow 0 \\
1 & \rightarrow 0 & \rightarrow 1 & \rightarrow 3 & \rightarrow 2 & \rightarrow 4 & \rightarrow 1
\end{align*}
\]
Every regular language can be described by a regular expression.

**G NFA**

\[ a, b, c, a, b, a, b, c, a, b b, c \]

Note: No loss in assuming no edges into \( q_0 \) out of \( F \) only one \( q_f \in F \)
GNFA

\[ G = (Q, \Sigma, \delta, q_0, q_f) \]

\( Q, \Sigma, q_0, q_f \in Q \) as usual

\[ S: (Q - \{ q_f \}) \times (Q - \{ q_0 \}) \rightarrow R \Sigma \]

**Definition**

- \( G \) can be in state \( q \in Q \) after reading
  \( x \in \Sigma^* \) if \( \exists k \geq 0, \)
  \( \exists r_0, r_1, \ldots, r_k \in Q \)
  \( \exists x_1, \ldots, x_k \in \Sigma^* \)
  such that
  \( x = x_1 \cdot x_2 \cdot \ldots \cdot x_k \)
  \( r_0 = q \)
  \( r_k = q_f \)
  \( \forall 1 \leq i \leq k, x_i \in L(\delta(r_{i-1}, r_i)) \)

- \( L(G) = \{ x \mid G \text{ can be in state } q_0 \ldots q_f \} \)

**Note:** The syntax is a little different; maps state pair to label (reg. exp.) rather than state x symbol = new state.
**Theorem**

If $L$ is accepted by a CNFA, then $L$ is regular.

**Proof Sketch:** Replace edge labeled "r" by NFA equivalent to $r$ based on previous theorem.