L = \{ w \in \{a,b\}^* \mid \text{3rd letter from the right end of } w \text{ is } "a" \} \\

M = (\emptyset, \Sigma, \delta, \epsilon, F) \\
\Sigma = \{a, b, \emptyset\} \\
\emptyset = \{w \in \Sigma^* \mid |w| \leq 3 \} \\
\delta_0 = \emptyset \\
F = \{w \in \Sigma^* \mid w = ax, |x| = 2 \} \\
\delta(w, c) = \text{last 3 letters of } w \text{ } \text{ } \text{shift regrets}
DEF

"M ends in state \( q \) after reading \( w = \varepsilon \) if

(1) \( w = w_1 w_2 \ldots w_n \)
where \( w_i \in \Sigma \)

(2) \( \exists \) state \( r_0, r_1, r_2 \ldots r_n \in Q \)

such that:
(a) \( r_0 = q_0 \)
(b) \( \forall 1 \leq i \leq n \)
\( \delta(r_{i-1}, w_i) = r_i \)

(c) \( r_n = q \)

Fact: \( q \) is unique
because \( \delta \) is a function, basically.
Defn

M accepts \( w \in \Sigma^* \) \( \iff \) the state, \( q \), reached by M after reading \( w \) is an accepting state, i.e., \( q \in F \).

Defn

The language recognized by M,

\[ L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \} \]

Note

Every M recognizes exactly one language. Implicitly, it "recognizes" both strings it must accept and those it must reject.
Example

\[ M: \begin{array}{c}
\end{array} \]

\[ L(M) = \Sigma^* \]

\[ L_{pal} = \{ w \in \Sigma^* \mid w = w^{-1} \} \]

e.g. 101 and 001100 are palindromes

110 is not

\[ M \text{ above accepts every palindrome} \]

\[ \therefore L_{pal} \subseteq L(M) \]

but \( M \) also accepts some (in fact, all) non palindromes

\[ \therefore L_{pal} \neq L(M) \]
Regular Languages

$L \subseteq \Sigma^*$ is regular iff $L = L(M)$ for some F.A. $M$

Examples

"even parity" is regular
"3rd from right" is regular
"odd length" is regular
"$\Sigma^*$" is regular

Are there general ways to prove languages are regular, other than making more & more example M's?
Theorem

If $L$ is regular then so is $\Sigma^* - L$.

Proof

$L$ regular, so $L = L(M)$ for some FA $M = (Q, \Sigma, \delta, q_0, F)$.

Let $M' = (Q, \Sigma, \delta, q_0, Q - F)$.

For all $w \in \Sigma^*$:

- $M$ accepts $w$ $\iff$
- $M$ is in a state $q \in F$ after reading $w$.
- $M'$ rejects $w$ (since $q \in F \iff q \notin Q - F$).

Hence, $w \in L(M) \iff w \notin L(M')$.

i.e., $L(M') = \Sigma^* - L$ is regular.
Closure Properties

A set is "closed" under some operation if applying the op to set members always yields a set member.

Examples

$\mathbb{N}$ is closed under $+ \times$ (eg $1+2 \in \mathbb{N}$)
but not under $-$ / (eg $1-2 \notin \mathbb{N}$)

$\mathbb{Z}$ is closed under $+ - \times$ (1-2$\notin \mathbb{Z}$)
but not under / (1/2 $\notin \mathbb{Z}$)

The set of regular language
is closed under complementation.