1. (20 points) Give examples of each of the following if possible. If not possible, explain why.
   a. Two countably infinite sets A and B such that A is a proper subset of B
   b. Two countably infinite sets whose cross product is uncountably infinite
   c. Two uncountably infinite sets whose intersection is finite
   d. Two uncountably infinite sets A and B such that (A-B) is uncountably infinite

2. (10 points) You are in the restroom of your local theatre that is playing the new disaster movie (or disaster of a movie) starring Ben Affleck. The restroom contains 6 stalls in a row. If 4 of these stalls are empty, prove that there is at least one empty stall that has another empty stall next to it. (Hint: Use the pigeonhole principle.)

3. (20 points) Consider the set \( \Sigma^* \) for \( \Sigma = \{0,1\} \).
   a. Prove that \( \Sigma^* \) is countably infinite.
   b. At the annual CSE 322 theorem-proving cocktail party, a party crasher announces the following “proof” by diagonalization that \( \Sigma^* \) is in fact uncountable. What is wrong with this “proof”?
      “Proof: By Contradiction. Suppose \( \Sigma^* \) is countably infinite. Then, there exists a bijection \( f: \mathbb{N} \rightarrow \Sigma^* \). We can create a new string \( s \) as follows:
      \( i \)th symbol of \( s \) =
      0 if the \( i \)th symbol of \( f(i) \) is 1
      1 if the \( i \)th symbol of \( f(i) \) is 0
      1 if length of \( f(i) \) < \( i \) (i.e. \( i \)th symbol does not exist)
      Then, \( s \) differs from all the strings given by the function \( f \). Since \( s \) is a binary string that is not among the outputs of \( f \), this contradicts the fact that \( f \) is a bijection. Therefore, \( \Sigma^* \) is uncountable.”

4. (50 points) Draw state diagrams of (deterministic) finite automata that recognize the following languages. In all cases, the alphabet is \{0,1\}.
   a. \{w | w begins with 1 and ends in 0\}
   b. \{w | number of 1’s in w is divisible by 3\}
   c. \{w | the third symbol of w is 1 and w has odd length\}
   d. \{w | each 1 in w is immediately preceded by a 0\}
   e. \{w | w contains an odd number of 0s and at least two 1s\}
   f. \{w | w contains a single 00 and a single 11\}
   g. \{w | w contains at least two 0s and at most four 1s\}
   h. \{w | w does not contain 101 or 111\}
   i. the set \{\varepsilon\}
   j. the empty set