Notes for Friday, April 30th
We have previously looked at these examples: $L_1 = \{0^n1^n | n \geq 0 \}$ and $L_2 = \{ww | w \in \{0,1\}^* \}$. Now we will look at more examples.

Example: show that $L_P = \{0^n | n \text{ is prime} \}$ is not regular. (Remark: this shows that a language with a single symbol in its alphabet doesn’t have to be simple)

Proof (by contradiction):

1. Assume $L_P$ is regular

2. There exists a $p$ as in the Pumping Lemma

3. Choose $s \in L_P$ such that $|s| \geq p$, so $s = 0^k$ with $k \text{ prime}$, $k \geq p + 2$ (we use $p + 2$ rather than $p$ to show a result later. We may always find such a $k$ because there are infinitely many primes.)

4. For any $x, y, z$ such that $s = xyz$, $|y| \geq 1$ and $|xy| \leq p$.

5. We need to choose $i$ such that $xy^iz \notin L_P$. What does this mean? We want $xy^iz = 0^m$ such that $m$ is not prime, or that $m = n_1 \times n_2$ with $n_1, n_2 \geq 2$. Hence, we want $|xy^iz| = |xz| + |y| = n_1n_2$ for such $n_1, n_2$.

   Choose $i = |xz|$. Then, $|xy^iz| = |xz| + |xz| |y| = |xz|(1 + |y|)$. We have $n_1 = |xz|$ and $n_2 = 1 + |y|$. Notice that $|y| \geq 1$, so $n_2 \geq 2$. Also, $|y| \leq p$ and $|xyz| \geq p + 2$ by the choice of $s$ and the assertions of the Pumping Lemma, so $|z| \geq 2$, or $|xz| = n_1 \geq 2$, as desired. We have shown that for our choice $i = |xz|$, $|xy^iz|$ is not prime and thus $s \notin L_P$, which is a contradiction.

There are some more useful tricks.

Example: Show that $L_3 = \{w | w \in \{0,1\}^* \text{ and } w \text{ contains an equal number of 0s and 1s} \}$ is not regular.

We could prove it like we proved $L_1$ is not regular, using the same string. Or, we could notice that $L_3 \cap 0^*1^* = L_1$. If $L_3$ were regular, by the closure properties of regular languages, the intersection of the two regular languages $L_3$ and $0^*1^*$ would be regular, but this intersection ($L_1$) is not regular, which is a contradiction. Hence, $L_3$ is not regular.

Example: (Language was also called “Distinct” in lecture): Show that $L = \{w | w = x_1\#x_2\#\cdots\#x_k, k \geq 0, x_i \in 1^*, x_i \neq x_j \text{ for } i \neq j \}$. (A collection of strings that have an unequal number of 1s)

Assume $L$ is regular. Then, $L \cup 1^*\#1^* = \{1^n\#1^n \}$, but this result is not regular (proof is similar to that for language $L_1$), so by the closure properties of languages this is a contradiction. Hence, $L$ is not regular.

In particular, we may use the that regular languages are closed under complement, intersection, union, concatenation, and star.