1. For the DFA below, prove for all \( i \in Q \) and \( w \in \Sigma^* \) that \( M \) is in state \( i \) after reading \( w \) if and only if \( \#_a(w) \equiv i \, (\text{mod} \, 5) \), where \( \#_a(w) \) is the number of \( a \)'s in the string \( w \). Prove that \( L(M) = \{ w \mid \#_a(w) \equiv 1 \, (\text{mod} \, 5) \text{ or } \#_a(w) \equiv 4 \, (\text{mod} \, 5) \} \).

2. Let \( \Sigma = \{0, 1\} \). For any string \( w \in \Sigma^* \), define a function from \( b : \Sigma^* \to \mathbb{N} \) (the natural numbers) as follows:

\[
b(w) = \begin{cases} 
0 & \text{if } w = \epsilon \\
2 \cdot b(x) + a & \text{if } w = xa \text{ for some } x \in \Sigma^* \text{ and } a \in \Sigma.
\end{cases}
\]

(In the second case of the definition, the “+a” is treating “a” as an integer in the obvious way, rather than as a character from \( \Sigma \).)

(a) What are \( b(1), b(10), b(100), b(1001), b(10011) \)? Say in simple English what the function \( b \) is. (I want a non-algorithmic description.)

(b) Let \( M = (Q, \Sigma, \delta, 0, \{0\}) \) where \( Q = \{0, 1, \ldots, 4\} \) and for all \( q \in Q, a \in \Sigma, \delta(q, a) = (2 \cdot q + a) \text{ mod } 5 \). Draw \( M \)'s state diagram.

(c) Give the sequences of states \( M \) is in while reading 10011.

(d) Give a concise, mathematical statement characterizing what state \( M \) is in after reading any given string \( w \). E.g., “\( M \) is in state \( i \) iff \( 42 \cdot \#_1(w) \equiv i \, (\text{mod} \, 5) \)” is an (incorrect) example of what such a statement might look like.

(e) Prove it, by induction on \( |w| \).

(f) Based on (d,e), what is \( L(M) \)?

3. (a) Give a formal inductive proof of the key claim needed to establish the correctness of the “Cartesian product construction” used in Theorem 1.25 (1st ed.: 1.12): For all \( x \in \Sigma^* \), and all \( r_1 \in Q_1, r_2 \in Q_2 \), we will have \( M \) in state \( (r_1, r_2) \) after reading \( x \) if and only if \( M_1 \) is in state \( r_1 \) after reading \( x \), for \( i = 1, 2 \).

(b) Then use this fact to prove that \( L(M) = L(M_1) \cup L(M_2) \).
(c) Modify the construction of $M$ slightly, giving a DFA $M'$ accepting $L(M_1) - L(M_2)$ (i.e., the set of strings in $L(M_1)$ but not in $L(M_2)$). Use part (a) to prove your construction correct (i.e., that $L(M') = L(M_1) - L(M_2)$).