Problems:

1. Give state diagrams of NFAs with the specified number of states recognizing each of the following languages. In each part the alphabet is \{0, 1\}.
   
   (a) The language \{0\}^* with one state. Recall \{0\}^* = \{\varepsilon, 0, 00, 000, \ldots\}.
   
   (b) The language \{\varepsilon\} with one state.
   
   (c) The language \{0\} with two states.
   
   (d) The language \{w | w = x0101y for some x, y \in \{0, 1\}^*\} with five states.

2. Given two strings \(x\) and \(y\) of exactly the same length, we can create a new string called \(\text{shuffle}(x, y)\) that consists of characters of \(x\) and \(y\) alternating one after another starting with the first character of \(x\). That is, if \(x = x_1 \ldots x_k\) and \(y = y_1 \ldots y_k\), then

   \[\text{shuffle}(x, y) = x_1y_1x_2y_2 \ldots x_ky_k\]  

   (1)

   For languages \(A\) and \(B\), define \(\text{SHUFFLE}(A, B) = \{\text{shuffle}(x, y) | x \in A, y \in B\ \text{and}\ |x| = |y|\}\). Given DFAs that accept \(A\) and \(B\), give an intuitive description and then a formal description of how to build a DFA that accepts \(\text{SHUFFLE}(A, B)\). (Note that in the above we did not specify that \(A\) and \(B\) have the same alphabet. Also note that the DFA for \(\text{SHUFFLE}(A, B)\) only gets one symbol at a time, that is, on input \(x_1y_1x_2y_2 \ldots x_ky_k\), it reads as \(x_1\) then \(y_1\) then \(x_2\), etc, and not (for example) in pairs like \(x_1y_1\) then \(x_2y_2\), etc.

3. Convert the following NFA to a DFA using the subset construction discussed in class. For your answer you should draw the state diagram for your DFA. (You may omit, if they exist, states which are not reachable from the start state.)

4. In this problem you will prove that regular languages are closed under certain operations. For all the three parts, assume \(L\) and \(M\) are regular languages. (If you prove
these by giving a construction of a DFA/NFA, present a correct construction of the DFA/NFA, including a formal description and also give an informal discussion yet convincing explanation of why the construction works. You do not need to give a formal proof of correctness using induction.)

(a) Prove that \( L^R = \{ x^R \mid x \in L \} \) is also regular. Recall that the \( R \) operation reverses the order of the string.

(b) Prove that \( \bar{L} = \{ x \in \Sigma^* \mid x \notin L \} \) is also regular. \( \bar{L} \) it the complement of \( L \) in \( \Sigma^* \).

(c) Prove that \( L - M = \{ w \mid w \in L \text{ but } w \notin M \} \) is also regular.

5. An odd-NFA \( M \) is a 5-tuple \((Q, \Sigma, \delta, s, F)\) that accepts a string \( x \in \Sigma^* \) if the number of possible states that \( M \) could be in after reading input \( x \), which are also in \( F \), is an odd number. (In other words, the set of all possible states has an odd number of states from \( F \).) Note, in contrast, a regular NFA accepts a string if some state among these possible states is a final state.

Prove that odd-NFAs accept the set of regular languages.

6. Extra Credit (do it for the glory, not the points!) Prove that if \( L \) is regular, then \( \text{half}(L) \) is regular, where the operation \( \text{half} \) is defined as follows:

\[
\text{half}(L) = \{ x \mid \text{for some } y \text{ such that } |x| = |y|, xy \text{ belongs to } L \} 
\] (2)