Defining $\delta^*$ from $\delta$

In the definition of DFAs, the transition function $\delta$ explicitly describes for each character $a \in \Sigma$, the name of the state reached on $a$ when started at state $q$. This is precisely $\delta(q, a)$.

In analyzing DFAs we often want to talk about the state that a given string $w \in \Sigma^*$ reaches when started at a state $q$. We give this corresponding function the name $\delta^*$; that is $\delta^*(q, w)$ is the state that would be reached starting at state $q$ and following the string $w \in \Sigma^*$. This function $\delta^*$ is determined entirely based on $\delta$ using the following inductive definition.

- $\delta^*(q, \varepsilon) = q$
- for $x \in \Sigma^*$ and $a \in \Sigma$, $\delta^*(q, xa) = \delta(\delta^*(q, x), a)$.

Note that this immediately means that $\delta^*(q, a) = \delta(\delta^*(q, \varepsilon), a) = \delta(q, a)$.

The following is a useful property of the $\delta^*$ function.

**Theorem 1.** For any $q \in Q$, and $x, y \in \Sigma^*$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.

**Proof.** The proof is by induction on the length of $y$ where the property we prove for each $y$ is that for all $x \in \Sigma^*$, for all $q \in Q$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.

**BASE CASE:** $y = \varepsilon$. In this case for any $x \in \Sigma^*$ and $q \in Q$,

$$\delta^*(q, xy) = \delta^*(q, x) = \delta^*(\delta^*(q, x), \varepsilon)$$

since $y = \varepsilon$ by the definition of $\delta^*$

**INDUCTION HYPOTHESIS:** Assume that for all $x \in \Sigma^*$, for all $q \in Q$, $\delta^*(q, xy) = \delta^*(\delta^*(q, x), y)$.

**INDUCTION STEP:** Let $y' = ya$ where $y \in \Sigma^*$ and $a \in \Sigma$. Then

$$\delta^*(q, xy') = \delta^*(q, xy)$$

by definition

$$= \delta(\delta^*(q, xy), a)$$

by the definition of $\delta^*$

$$= \delta(\delta^*(\delta^*(q, x), y), a)$$

by the inductive hypothesis

$$= \delta(\delta^*(p, y), a)$$

where $p = \delta^*(q, x)$

$$= \delta^*(p, ya)$$

by the definition of $\delta^*$

$$= \delta^*(\delta^*(q, x), ya)$$

by the definition of $p$

$$= \delta^*(\delta^*(q, x), y')$$

which is what we needed to prove. Therefore by induction on the length of $y$ the claim is proved. \qed