Diagonalization

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May 31

Announcements

- Pick up solutions to H/W #7
- Pick up old graded H/Ws
- Official feedback at the end of class today
- Reminders
  - Final exam
    - Monday, June 5, 2:30-4:20p, here
  - Review Session
    - Sunday, June 4th, 3:00-4:00p, TBA

Today’s puzzle

- Prove that $\Sigma^*$ is countable
  - $\Sigma$ is fixed, say $\{0,1\}$
- Prove that the set of all languages $L \subseteq \Sigma^*$ is uncountable

Last lecture

- A set $B$ is countable if
  - There exists a function $f : \mathbb{N} \to B$
  - $f$ is one to one and onto

In an alternate matrix…

The set of real number is uncountable

Really? I do not believe you

But Morpheus knows Cantor…
Cantor’s proof

- For contradiction, assume \( \mathbb{R} \) is countable
- Exists a one to one onto function \( f : \mathbb{N} \to \mathbb{R} \)
- List for each \( i \), the real \( f(i) \)
- “Construct” a real \( r \)
  - There no \( i \) such that \( r = f(i) \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>( f(i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0012345...</td>
</tr>
<tr>
<td>2</td>
<td>0.0081909...</td>
</tr>
<tr>
<td>3</td>
<td>0.1000000...</td>
</tr>
<tr>
<td>4</td>
<td>0.00056792...</td>
</tr>
</tbody>
</table>

Construction of the absent \( r \)

- \( r = 0.r_1r_2r_3r_4\ldots \)
- \( r_i \neq i \)th digit of \( f(i) \)
- For every \( i \), \( r \neq f(i) \)
  - Contradiction as \( f \) is onto

Diagonalization because the proof “changes” the diagonal

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<th>( r )</th>
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<tbody>
<tr>
<td>1</td>
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<td>0.00012345...</td>
</tr>
<tr>
<td>2</td>
<td>0.0081909...</td>
<td>0.00820093...</td>
</tr>
<tr>
<td>3</td>
<td>0.1000000...</td>
<td>0.10000009...</td>
</tr>
<tr>
<td>4</td>
<td>0.00056792...</td>
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</tr>
</tbody>
</table>

Up next...

- \( A_{TM} = \{ \langle M,w \rangle \mid M \text{ is a TM that accepts } w \} \)
- This language is undecidable
  - Proof by diagonalization
  - First “version” might seem like “magic”
  - “Deciding if a TM is a decider is undecidable”

Feedback

- If you are writing comments on the yellow sheet
- It would be great if in the improvement section, mention
  - Anything you think is important
  - Coverage of topics
    - Something you wish you had seen more/less of?
  - Puzzles
  - Neighbor talk time