Correctness Proof for Theorem 1.39 [1st Ed: Theorem 1.19]
CSE 322: Introduction to Formal Models in Computer Science
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There is nothing “obvious” about the construction in the proof of Theorem 1.39, so the statement near the end of the proof that “the construction of \( M \) obviously works correctly” is obviously incorrect. Here is a proof.

Let \( A = (Q, \Sigma, \delta, q_0, F) \) be any finite automaton (either deterministic or nondeterministic), \( p, q \in Q \), and \( x, y \in \Sigma^* \). The notation \( (p, xy) \trans_A^*(q, y) \) means that, if you start \( A \) in state \( p \) with input \( xy \), then in zero or more transitions \( A \) can get to state \( q \) with input \( y \) remaining unread (that is, \( A \) can get to state \( q \) after consuming just the prefix \( x \)). The notation \( \trans_A \) without the * is analogous, but is used to indicate that the move from \( p \) to \( q \) happens after exactly one transition rather than in zero or more transitions.

**Lemma 1** \((q_0, w) \trans_N^*(r, \varepsilon) \iff (q'_0, w) \trans_M^*(R, \varepsilon) \) and \( r \in R \).

**Proof:** The proof is by induction on \(|w|\).

**Basis** \((w = \varepsilon)\):

\[(q_0, \varepsilon) \trans_N^*(r, \varepsilon) \iff r \in E(\{q_0\}) \quad \text{(defn of } E)\]

\[\iff q'_0 = R \text{ and } r \in R \quad \text{(defn of } q'_0)\]

\[\iff (q'_0, \varepsilon) \trans_M^*(R, \varepsilon) \text{ and } r \in R \quad \text{(no } \varepsilon \text{ transitions)}\]

**Induction** \((w = xa)\):

\[(q_0, xa) \trans_N^*(r, \varepsilon) \]

\[\iff (\exists s, t) \quad (q_0, xa) \trans_N^*(s, a) \text{ and } (s, a) \trans_N(t, \varepsilon) \text{ and } (t, \varepsilon) \trans_N^*(r, \varepsilon)\]

\[\iff (\exists s, t) \quad (q_0, x) \trans_N^*(s, \varepsilon) \text{ and } t \in \delta(s, a) \text{ and } r \in E(\{t\}) \quad \text{(defn of } E)\]

\[\iff (\exists s, t) \quad (q'_0, x) \trans_M^*(S, \varepsilon) \text{ and } s \in S \text{ and } t \in \delta(s, a) \text{ and } r \in E(\{t\}) \text{(Ind Hyp)}\]
iff \((q'_0, x) \vdash_M^* (S, \varepsilon)\) and \(r \in \bigcup_{s \in S} E(\delta(s, a))\)

iff \((q'_0, x) \vdash_M^* (S, \varepsilon)\) and \(r \in \delta'(S, a)\) \hspace{1cm} \text{(defn of} \ \delta')

iff \((q'_0, xa) \vdash_M^* (S, a)\) and \(\delta'(S, a) = R\) and \(r \in R\)

iff \((q'_0, xa) \vdash_M^* (S, a)\) and \((S, a) \vdash_M (R, \varepsilon)\) and \(r \in R\)

iff \((q'_0, xa) \vdash_M^* (R, \varepsilon)\) and \(r \in R\)

\(\square\)

Now we can use this lemma to prove the correctness of the construction in Theorem 1.39.

**Theorem 2** \(L(M) = L(N)\).

**Proof:**

\(w \in L(M)\) iff \((q'_0, w) \vdash_M^* (R, \varepsilon)\) and \(R \in F'\) \hspace{1cm} \text{(defn of} \ L(M))

iff \((q'_0, w) \vdash_M^* (R, \varepsilon)\) and \(R \cap F \neq \emptyset\) \hspace{1cm} \text{(defn of} \ F')

iff \((\exists r) \ (q'_0, w) \vdash_M^* (R, \varepsilon)\) and \(r \in R\) and \(r \in F\)

iff \((\exists r) \ (q_0, w) \vdash_N^* (r, \varepsilon)\) and \(r \in F\) \hspace{1cm} \text{(Lemma 1)}

iff \(w \in L(N)\) \hspace{1cm} \text{(defn of} \ L(N))

\(\square\)