It is intuitively clear that the machine presented in the proof of Theorem 1.12 recognizes the language $A_1 \cup A_2$. Informally speaking, the first component of the state in the machine $M$ changes just as though machine $M_1$ were running independently of $M_2$. The second component of the state in the machine $M$ changes just as though machine $M_2$ were running independently of $M_1$. $M$ accepts $w$ if either machine ends up in a final (accept) state.

Here is a proof that formalizes this intuition.

**Lemma 1** \( L(M) = A_1 \cup A_2 \).

**Proof:**

\[
\begin{align*}
w \in L(M) & \iff ((q_1,q_2),w) \vdash^*_M ((f_1,f_2),\varepsilon), \text{ where } f_1 \in F_1 \text{ or } f_2 \in F_2. \\
& \iff (q_1,w) \vdash^*_M (f_1,\varepsilon) \text{ and } (q_2,w) \vdash^*_M (f_2,\varepsilon), \text{ where } f_1 \in F_1 \text{ or } f_2 \in F_2. \\
& \iff w \in L(M_1) \text{ or } w \in L(M_2) \\
& \iff w \in L(M_1) \cup L(M_2)
\end{align*}
\]

\( \square \)

Technically speaking, in order to get from line (1) to line (2) above, we would have to prove by induction on the length of the string $w$ that for any states $s_1, p_1 \in Q_1$, $s_2, p_2 \in Q_2$, and any string $w$, \(((s_1,s_2),w) \vdash^*_M ((p_1,p_2),\varepsilon) \iff (s_1,w) \vdash^*_M (s_1,\varepsilon) \text{ and } (s_2,w) \vdash^*_M (p_2,\varepsilon)\). (Do you see why we can’t conclude (2) from (1) directly?)

**Claim:** For any states $s_1, p_1 \in Q_1$, $s_2, p_2 \in Q_2$, and any string $w$, \(((s_1,s_2),w) \vdash^*_M ((p_1,p_2),\varepsilon) \iff (s_1,w) \vdash^*_M (p_1,\varepsilon) \text{ and } (s_2,w) \vdash^*_M (p_2,\varepsilon)\).

**Proof:** The proof is by induction on $|w|$.

**Basis:** Left as an exercise.

**Induction Hypothesis:** Suppose the Claim is true for all strings $z$ such that $0 \leq |z| \leq n$, for some fixed $n \geq 0$.

**Induction Step:** Let $w$ be a string where $|w| = n + 1$ and $w = w'a$ for some $a \in \Sigma$. 

1
There are two implications to prove.

(⇒) Suppose \(((s_1, s_2), w'a) \vdash_M^* ((p_1, p_2), \varepsilon)\).

Then \(((s_1, s_2), w'a) \vdash_M^* ((r_1, r_2), a) \vdash_M ((p_1, p_2), \varepsilon)\), where \(r_1 \in Q_1, r_2 \in Q_2, \delta_1(r_1, a) = p_1,\)
and \(\delta_2(r_2, a) = p_2\).

By Fact 1, \(((s_1, s_2), w') \vdash_M^* ((r_1, r_2), \varepsilon)\).

By the induction hypothesis, \((s_1, w') \vdash_{M_1}^* (r_1, \varepsilon)\) and \((s_2, w') \vdash_{M_2}^* (r_2, \varepsilon)\).

So,

\[
(s_1, w') \vdash_{M_1}^* (r_1, a) \quad \text{(Fact 1)}
\]

\[
\vdash_{M_1} (p_1, \varepsilon) \quad (\delta_1(r_1, a) = p_1)
\]

and

\[
(s_2, w') \vdash_{M_2}^* (r_2, a) \quad \text{(Fact 1)}
\]

\[
\vdash_{M_2} (p_2, \varepsilon) \quad (\delta_2(r_2, a) = p_2)
\]

(⇐) This implication direction is left as an exercise.

\[\square\]