Recap of Undecidability Proof

- **The Question**: Are there languages that are not decidable by any Turing machine (TM)?
  - i.e. Are there problems that cannot be solved by any algorithm?
- Consider the language:
  \[ A_{TM} = \{ <M,w> \mid M \text{ is a TM and } M \text{ accepts } w \} \]
  (Recall that \(<A,B,…>\) is just a string encoding the objects A, B, …)
- What can we say about \(A_{TM}\)?

\[ A_{TM} \text{ is Turing-recognizable} \]

- \( A_{TM} \text{ is Turing-recognizable: Recognizer TM } R \text{ for } A_{TM}: \)
  - On input string \(<M,w>\):
    - Simulate \(M\) on \(w\).
    - ACCEPT \(<M,w>\) if \(M\) halts & accepts \(w\);
    - REJECT \(<M,w>\) if \(M\) halts & rejects
      (Loop (& thus reject \(<M,w>\)) if \(M\) ends up looping).
  - \(R\) accepts \(<M,w>\) iff \(M\) accepts \(w\), i.e. \(L(R) = A_{TM}\)

Yeah, but is it decidable?!!
Is $A_{TM}$ decidable?

- No, $A_{TM} = \{<M,w>| M \text{ is a TM and } M \text{ accepts } w\}$ is undecidable! 1-slide Proof (by Contradiction):
  1. Assume $A_{TM}$ is decidable $\Rightarrow$ there’s a decider $H$, $L(H) = A_{TM}$
  2. $H$ on $<M,w> = \text{ACC}$ if $M$ accepts $w$
     \hspace{1cm} $= \text{REJ}$ if $M$ rejects $w$ (halts in $q_{\text{REJ}}$ or loops on $w$)
  3. Construct new TM $D$: On input $<M>$:
     - Simulate $H$ on $<M,<M>>$ (here, $w = <M>$)
     - If $H$ accepts, then REJ input $<M>$
     - If $H$ rejects, then ACC input $<M>$
  4. What happens when $D$ gets $<D>$ as input?
    - $D$ rejects $<D>$ if $H$ accepts $<D,<D>>$ if $D$ accepts $<D>$
    - $D$ accepts $<D>$ if $H$ rejects $<D,<D>>$ if $D$ rejects $<D>$
  Either way: Contradiction! $D$ cannot exist $\Rightarrow H$ cannot exist
  Therefore, $A_{TM}$ is not a decidable language.

Undecidability Proof uses Diagonalization:

Input strings $<M_1> <M_2> <M_3> \ldots$ $<M_1> <M_2> <M_3> \ldots <D>$

<table>
<thead>
<tr>
<th>List of TMs</th>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>ACC</td>
<td>REJ</td>
<td>loop</td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td>REJ</td>
<td>loop</td>
<td>ACC</td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
<td>ACC</td>
<td>ACC</td>
<td>REJ</td>
<td></td>
</tr>
</tbody>
</table>

If $H$ exists $D$ outputs opposite of diagonal

$D$ on $<M_i>$ accepts if and only if $M_i$ on $<M_i>$ rejects.
So, $D$ on $<D>$ will accept if and only if $D$ on $<D>$ rejects!
A contradiction $\Rightarrow H$ cannot exist!
Therefore, $A_{TM}$ is not a decidable language.
One Last Concept: Reducibility

✦ How do we show a new problem B is undecidable?

✦ Idea: Show that \( A_{TM} \) is reducible to the new problem B
  ➔ What does this mean and how do we show this?

✦ Show that if B was decidable, then you can use the decider for B as a subroutine to decide \( A_{TM} \)
  ➔ Contradiction, therefore B must also be undecidable

The Halting Problem is Undecidable (Turing, 1936)

✦ Halting Problem: Does TM M halt on input w?
  ➔ Equivalent language: \( A_H = \{ <M,w> \mid \text{TM M halts on input w} \} \)
  ➔ Need to show \( A_H \) is undecidable
  ➔ We know \( A_{TM} = \{ <M,w> \mid \text{TM M accepts w} \} \) is undecidable

✦ Show \( A_{TM} \) is reducible to \( A_H \) (Theorem 5.1 in text)
  ➔ Suppose \( A_H \) is decidable \( \Rightarrow \) there’s a decider \( M_H \) for \( A_H \)
  ➔ Then, we can construct a decider \( D_{TM} \) for \( A_{TM} \):
    On input \( <M,w> \), run \( M_H \) on \( <M,w> \).
    - If \( M_H \) rejects, then REJ (this takes care of M looping on w)
    - If \( M_H \) accepts, then simulate M on w until M halts
    - If M accepts, then ACC input \( <M,w> \); else REJ
    \( L(D_{TM}) = A_{TM} \Rightarrow A_{TM} \) is decidable! Contradiction \( \Rightarrow A_H \) is undecidable

✦ E.g. 2: Show \( E_{TM} = \{ <M> \mid \text{M is a TM and } L(M) = \emptyset \} \) is undecidable (see Theorem 5.2 in the text)
Are There Languages That Are Not Even Recognizable?

- \( A_{TM} \) and \( A_H \) are undecidable but Turing-recognizable
  - Are there languages that are not even Turing-recognizable?

- What happens if both \( A \) and \( \overline{A} \) are Turing-recognizable?
  - There exist TMs \( M1 \) and \( M2 \) that recognize \( A \) and \( \overline{A} \)
  - **Can construct a decider for \( A \)**
    1. On input \( w \):
       1. Simulate \( M1 \) and \( M2 \) on \( w \) one step at a time, alternating between them.
       2. If \( M1 \) accepts, then ACC \( w \) and halt; if \( M2 \) accepts, REJ \( w \) and halt.

- \( A \) and \( \overline{A} \) are both Turing-recognizable iff \( A \) is decidable

- **Corollary:** \( \overline{A}_{TM} \) and \( \overline{A}_H \) are not Turing-recognizable
  - If they were, then \( A_{TM} \) and \( A_H \) would be decidable

The Chomsky Hierarchy of Languages

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<table>
<thead>
<tr>
<th>Language</th>
<th>Regular</th>
<th>Context-Free</th>
<th>Decidable</th>
<th>Turing-Recognizable</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Computational Models</strong></td>
<td>DFA, NFA, RegExp</td>
<td>PDA, CFG</td>
<td>Deciders – TMs that halt for all inputs</td>
<td>TMs that may loop for strings not in language</td>
</tr>
<tr>
<td><strong>Examples</strong></td>
<td>((0 \cup 1)^*11)</td>
<td>({0^n1^n \mid n \geq 0}), Palindromes</td>
<td>({0^n1^n0^n \mid n \geq 0}), (A_{DFA}), (A_{CFG})</td>
<td>(A_{TM}), (A_H)</td>
</tr>
</tbody>
</table>

(Chomsky also studied context-sensitive languages (CSLs, e.g. \(a^n b^n c^n\)) , a subset of decidable languages recognized by linear-bounded automata (LBA))
The Chomsky Hierarchy – Then & Now…

Then (1950s)

Not T-recognizable

\( \overline{A_{TM}} \)

\( A_{TM} \)

T-recognizable

Decidable

CFLs

\( 0^n1^n0^n \)

\( 0^n1^n \)

REG

\( 0^*1^* \)

U.S. interventionism in the developing world

Political economy of human rights

Propaganda role of corporate media

Now

Final Exam

- Details regarding the Final Exam
  - When: Monday, Dec. 16, 2002 from 8:30-10:20 a.m.
  - Where: This classroom EE1 037.
  - What will it cover?
    - Chapters 0-4 and Chapter 5: pages 171-176.
    - Emphasis will be on material covered after midterm (Chapter 2 and beyond)
    - You may bring 1 page of notes (8 ½” x 11” sheet!)
    - Approximately 6 questions
  - How do I ace it?
    - Practice, practice, practice!
    - See class website for sample final exam and solutions
I believe the Final exam is decidable!

I believe the world’s problems are politically decidable.

I believe my next movie will be unrecognizable.

Stay cool ‘n’ keep pumpin’!