Expectation

The expected value of random variable $X(s)$ on sample space $S$ is:

$$E(X) = \sum_{s \in S} p(s)X(s)$$

$$E(X) = \sum_{r \in X(S)} p(X = r)r$$
Flip a coin until the first head
Expected number of flips?

Random variable:

Computing the expectation:
Linearity of Expectation

$E(X_1 + X_2) = E(X_1) + E(X_2)$

$E(aX) = aE(X)$
Hashing

\[ H: M \rightarrow [0..n-1] \]

If \( k \) elements have been hashed to random locations, what is the expected number of elements in bucket \( j \)?

What is the expected number of collisions when hashing \( k \) elements to random locations?
Hashing analysis

Sample space: \([0..n-1] \times [0..n-1] \times \ldots \times [0..n-1]\)

Random Variables
\(X_j = \text{number of elements hashed to bucket } j\)
\(C = \text{total number of collisions}\)

\(B_{ij} = 1 \text{ if element } i \text{ hashed to bucket } j\)
\(B_{ij} = 0 \text{ if element } i \text{ is not hashed to bucket } j\)

\(C_{ab} = 1 \text{ if element } a \text{ is hashed to the same bucket as element } b\)
\(C_{ab} = 0 \text{ if element } a \text{ is hashed to a different bucket than element } b\)
Counting inversions

Let $p_1, p_2, \ldots, p_n$ be a permutation of $1 \ldots n$
$p_i, p_j$ is an inversion if $i < j$ and $p_i > p_j$

4, 2, 5, 1, 3

1, 6, 4, 3, 2, 5

7, 6, 5, 4, 3, 2, 1
Expected number of inversions for a random permutation

For each \( i < j \) let \( X_{ij} \) be the following random variable:

\[
X_{ij} = \begin{cases} 
1 & \text{if } p_i > p_j, \\
0 & \text{if } p_i < p_j
\end{cases}
\]

**Fact 1:** \( P(X_{ij} = 1) = \frac{1}{2} \) (why ??); hence \( E(X_{ij}) = \frac{1}{2} \)

Let \( X = \) the number of permutations: \( X = \sum_{ij} X_{ij} \)

**Fact 2:** \( E(X) = \sum_{ij} E(X_{ij}); \text{ hence } E(X) = \frac{n(n-1)}{4} \)
Insertion sort

for i :=1 to n-1{
    j := i;
    while (j > 0 and A[j - 1] > A[j]){
        swap(A[j -1], A[j]);
        j := j – 1;
    }
}

What is the expected number of swaps?
Expected number of swaps for Insertion Sort

For each $i = 1, \ldots, n-1$, let $X_i$ be the following random variable:

$$X_i = \text{number of swaps at iteration } i$$

**Fact 1:** For all $j = 0, \ldots, i$, $P(X_i = j) = 1/(i+1)$ \(\text{(why ??)}\)

**Fact 2:** $E(X_i) = (1+2+\ldots+i)/(i+1) = i/2$

Let $X$ be the total number of swaps

**Fact 3:** $E(X) = \sum_i E(X_i) = n(n+1)/4$
Left to right maxima

```plaintext
max_so_far := A[0];
k:=0;
for i := 1 to n-1
    if (A[i] > max_so_far)
        { max_so_far := A[i];
          k++;
        }
return k;
```

5, 2, 9, 14, 11, 18, 7, 16, 1, 20, 3, 19, 10, 15, 4, 6, 17, 12, 8

What is the expected value of k?
What is the expected number of left-to-right maxima in a random permutation