Announcements

- **Readings**
  - This week:
    - 6th edition: 4.3, 4.4, 5.1, 5.2
    - 5th edition: 3.4, 3.5, 4.1, 4.2
- **Midterm:**
  - Mean 67, Median 68

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Induction Example (revisited)

- Given a set $S$ of $n+1$ positive integers, none exceeding $2n$, show that $S$ is divisible.
- Paul Beame’s proof
  - Let $S \subseteq \{1, \ldots, 2n\}$ be non-divisible
  - Every element in $S$ can be written as $m^2i$ where $m$ is odd
  - We cannot have $m^2i$ and $m^2j$ both in $S$
  - Hence $|S| \leq n$

Highlights from Lecture 14

- **Recursive Definitions**
  - $F(0) = 1; F(n+1) = 2F(n)$
  - $f_0 = 0; f_1 = 1; f_n = f_{n-1} + f_{n-2}$

Recursive Definitions of Sets

- **Recursive definition**
  - Basis step: $0 \in S$
  - Recursive step: if $x \in S$, then $x + 2 \in S$
  - Exclusion rule: Every element in $S$ follows from basis steps and a finite number of recursive steps

Recursive definitions of sets

Basis: $6 \in S; 15 \in S$
Recursive: if $x, y \in S$, then $x + y \in S$

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$
Recursive:
- if $[x, y, z] \in S, \alpha \in R$, then $[\alpha x, \alpha y, \alpha z] \in S$
- if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$
  - then $[x_1 + x_2, y_1 + y_2, z_1 + z_2]$

Powers of 3
Strings

- The set $\Sigma^*$ of strings over the alphabet $\Sigma$ is defined
  - Basis: $\lambda \in \Sigma$ (the empty string)
  - Recursive: if $w \in \Sigma^*$, $x \in \Sigma$, then $wx \in \Sigma^*$

Families of strings over $\Sigma = \{a, b\}$

- $L_1$
  - $\lambda \in L_1$
  - $w \in L_1$ then $awb \in L_1$

- $L_2$
  - $\lambda \in L_2$
  - $w \in L_2$ then $aw \in L_2$
  - $w \in L_2$ then $wb \in L_2$

Function definitions

- $\text{Len}(\lambda) = 0$
- $\text{Len}(wx) = 1 + \text{Len}(w)$ for $w \in \Sigma^*$, $x \in \Sigma$

- $\text{Concat}(w, \lambda) = w$ for $w \in \Sigma^*$
- $\text{Concat}(w_1, w_2x) = \text{Concat}(w_1, w_2)x$ for $w_1, w_2$ in $\Sigma^*$, $x \in \Sigma$

Well Formed Formulae

- Basis Step
  - $T$, $F$, and $s$, where $s$ is a propositional variable are in WFF

- Recursive Step
  - If $E$ and $F$ are in WFF then $(\neg E)$, $(E \land F)$,
    $(E \lor F)$, $(E \rightarrow F)$ and $(E \leftrightarrow F)$ are in WFF

Tree definitions

- A single vertex $r$ is a tree with root $r$.
- Let $t_1$, $t_2$, $\ldots$, $t_n$ be trees with roots $r_1$, $r_2$, $\ldots$, $r_n$ respectively, and let $r$ be a vertex. A new tree with root $r$ is formed by adding edges from $r$ to $r_1$, $\ldots$, $r_n$.

Extended Binary Trees

- The empty tree is a binary tree.
- Let $r$ be a node, and $T_1$ and $T_2$ binary trees. A binary tree can be formed with $T_1$ as the left subtree and $T_2$ as the right subtree. If $T_1$ is non-empty, there is an edge from the root of $T_1$ to $r$. Similarly, if $T_2$ is non-empty, there is an edge from the root of $T_2$ to $r$. 
Full binary trees

• The vertex \( r \) is a FBT.
• If \( r \) is a vertex, \( T_1 \) a FBT with root \( r_1 \) and \( T_2 \) a FBT with root \( r_2 \) then a FBT can be formed with root \( r \) and left subtree \( T_1 \) and right subtree \( T_2 \) with edges \( r \to r_1 \) and \( r \to r_2 \).

Simplifying notation

• \((\bullet, T_1, T_2)\), tree with left subtree \( T_1 \) and right subtree \( T_2 \)
• \(\epsilon\) is the empty tree
• Extended Binary Trees (EBT)
  – \(\epsilon \in \text{EBT}\)
  – if \( T_1, T_2 \in \text{EBT} \), then \((\bullet, T_1, T_2) \in \text{EBT}\)
• Full Binary Trees (FBT)
  – \(\epsilon \in \text{FBT}\)
  – if \( T_1, T_2 \in \text{FBT} \), then \((\bullet, T_1, T_2) \in \text{FBT}\)

Recursive Functions on Trees

• \(N(T)\) - number of vertices of \( T \)
• \(N(\epsilon) = 0; N(\bullet) = 1\)
• \(N(\bullet, T_1, T_2) = 1 + N(T_1) + N(T_2)\)

• \(Ht(T)\) – height of \( T \)
• \(Ht(\epsilon) = 0; Ht(\bullet) = 1\)
• \(Ht(\bullet, T_1, T_2) = 1 + \max(Ht(T_1), Ht(T_2))\)

More tree definitions: Fully balanced binary trees

• \(\epsilon\) is a FBBT.
• if \( T_1 \) and \( T_2 \) are FBBTs, with \( Ht(T_1) = Ht(T_2) \), then \((\bullet, T_1, T_2)\) is a FBBT.

And more trees: Almost balanced trees

• \(\epsilon\) is a ABT.
• if \( T_1 \) and \( T_2 \) are ABTs with \( Ht(T_1)-1 \leq Ht(T_2) \leq Ht(T_1)+1 \) then \((\bullet, T_1, T_2)\) is a ABT.

Structural Induction

• Show \( P \) holds for all basis elements of \( S \).
• Show that \( P \) holds for elements used to construct a new element of \( S \), then \( P \) holds for the new elements.
**Prove all elements of S are divisible by 3**

- **Basis:** $6 \in S; \ 15 \in S$
- **Recursive:** if $x, y \in S$, then $x + y \in S$

**Prove that WFFs have the same number of left parentheses as right parentheses**