There are SEVEN problems, each worth 10 points.


1. For the relation $R = \{(b, c), (b, e), (c, e), (d, a), (e, b), (e, c)\}$ on $\{a, b, c, d, e\}$, draw the following relations in digraph form:
   (a) The reflexive closure of $R$.
   (b) The symmetric closure of $R$.
   (c) The transitive closure of $R$.
   (d) The reflexive, symmetric, transitive closure of $R$.

2. Let $R$ be a symmetric relation. Show that $R^n$ is symmetric for all positive integers $n$.

3. Let $p$ be an odd prime, and let $A = \{1, 2, \ldots, p-1\}$. Define a relation $T$ on $A$ by $T = \{(a, b) \in A \times A \mid a \equiv 2^s b \pmod{p} \text{ for some non-negative integer } s\}$. Prove that $T$ is an equivalence relation.

4. A relation $R$ is called circular if $a R b$ and $b R c$ imply that $c R a$. Show that $R$ is reflexive and circular if and only if it is an equivalence relation.

5. Consider a tournament among $n$ players where each player plays every other player exactly once and all the $\binom{n}{2}$ games thus played result in a win/loss result. Define a natural “defeat” relation $D$ on the set of players as follows: $a D b$ if player $a$ defeated player $b$ in their head-to-head encounter. Prove that the relation $D$ has the following property: there exists a player $w$ such that for every other player $x$ either $w D x$ or there exists a player $z$ such that $w D z$ and $z D x$. (Hint: Try for the candidate $w$ a player who has defeated the most other players.)

6. Prove that any simple, undirected graph on $n \geq 2$ vertices contains two vertices of equal degree.

7. Suppose $G$ is a simple, undirected graph on $2n$ vertices that contains no triangles (cycles of length 3). Prove that $G$ has at most $n^2$ edges. (Hint: Induction on $n$ is one possible approach.)