CSE 321: Discrete Structures Spring 2007

Problem Set 7

Due Friday, May 25, 2007, in class

There are SEVEN problems, each worth 10 points.

The exercise numbers refer to the number in Rosen's book, 6th Edition. When a different number is used in the 5th edition, that number is also mentioned. All problems are worth a total of 10 points unless mentioned otherwise.

Reading Assignment: 6th Edition: 6.3, 6.4, 8.1-8.3. 5th edition: 5.3, 5.4, 7.1-7.3.

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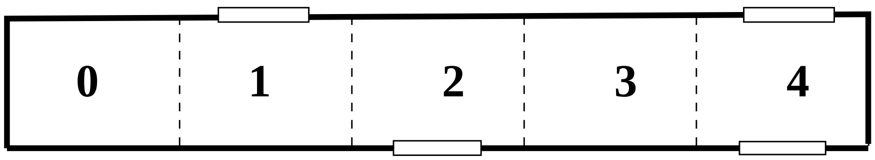
1. (a) Prove or disprove: If events E and F are independent, then so are $\overline{E}$ and $\overline{F}$.
   (b) Let A, B be events from a sample space with $\mathbf{Pr}(A) \neq 0$ and $\mathbf{Pr}(B) \neq 0$. Prove or disprove: If A makes B less likely, i.e., $\mathbf{Pr}(B|A) < \mathbf{Pr}(B)$, then B makes A less likely, i.e., $\mathbf{Pr}(A|B) < \mathbf{Pr}(A)$.
   (c) Prove or disprove: For a random variable X, $\mathbf{E}(X^2) = \mathbf{E}(X)^2$.

2. Suppose that n balls are tossed into b bins so that each ball is equally likely to fall into any of the bins and that the tosses are independent.
   (a) Find the probability that a particular ball lands in a specified bin.
   (b) What is the expected number of balls that land in a particular bin?
   (c) What is the expected number of balls tossed until a particular bin contains a ball?
   (d) What is the expected number of balls tossed until all bins contain a ball? (Hint: Define a random variable X_i equal to the number of tosses required to have a ball land in an i'th bin once $i-1$ bins contain a ball, and consider $\mathbf{E}(X_i)$.)

3. Great Expectations:
   (a) Suppose two fair 6-sided dice are rolled independently. Let Y be the random variable which is the absolute value of the difference of the two values showing. What is the expectation of Y? Let Z be the random variable which is the maximum of the two values showing. What is the expected value of Z?
   (b) (A 2006 Putnam problem) Let $S = \{1, 2, \ldots, n\}$ for some $n > 1$, Say that a permutation has a local maximum at $k \in S$ if
      i. $\pi(k) > \pi(k+1)$ for $k = 1$;
      ii. $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;
      iii. $\pi(k-1) < \pi(k)$ for $k = n$.

      (For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 1 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

4. *Mobile robot localization*: Bayes' rule underlies all modern AI systems for probabilistic inference. One application of this rule is the update of a robot's position estimate based on new sensor information. To see, look at the example below.



The robot is placed in the hallway facing east and it does not know where it is (it only knows its orientation). The hallway is tessellated and the robot can be in any of the five squares (always facing east). Assume that the width of each square is 1 meter. Then the squares are numbered by their distance from the beginning of the hallway. Given the tessellation of the hallway, we can represent the position of the robot by a random variable L. This variable takes on values between 0 and 4, depending on the robot's current position.

At each point in time, the robot's position estimate is represented by a probability distribution over all possible locations. In the beginning, the robot does not know where it is. This can be represented by the following distribution:

$$P(L = 0) = P(L = 1) = P(L = 2) = P(L = 3) = P(L = 4) = 1/5 \tag{1}$$

The robot has a camera that can be pointed to the left or to the right. In order to determine where it is, the robot tries to detect doors by looking either to the left or to the right (note that the robot faces east). As you can see in the figure, there are four doors in the hallway.

Now we want to update the robot's position estimate based on an observation, i.e. we want to compute the probability distribution of L *given* an observation. In general, we denote observations by o (where o ranges over the possible observations). The probability distribution *after* the observation can be represented by $P(L = l|o)$ (l is a number between 0 and 4). However, it is not easy to compute this value directly. In mobile robot localization we use Bayes' rule to compute this value:

$$P(L = l|o) \quad = \quad \frac{P(o \mid L = l)P(L = l)}{\sum_{i=1}^{n} P(o \mid L = l_i)P(L = l_i)} \tag{2}$$

All we need to know are the values for $P(o \mid L = l)$ and $P(L = l)$. $P(L = l)$ is given by the distribution *before* the robot made the observation (given in Equation (1)).

The term $P(o \mid L = l)$ describes the probability of making an observation *given* the position of the robot. In our example, we have four possible observations: The robot can detect a door to its left, it can detect a door to its right, it can detect a wall to its left, or it can detect a wall to its right. We will denote these four observations by DL, DR, WL, and WR (i.e. these are the four possible values of o). To determine $P(o \mid L = l)$, the probability of making observation o at location l, we make the following assumption about the robot's ability to detect doors and walls. If the robot points the camera towards a door, then it successfully detects this door with probability 0.7. However, it sometimes confuses the door for a wall, i.e. with probability 0.3 it erroneously detects a wall, even though it looks at a door. Detecting walls is easier, therefore, the probability of detecting a wall if the robot is looking at a wall is 0.9. With probability 0.1 it thinks it detects a door even though it looks at a wall. There are four different types of positions in the hallway: Let l_{DL} denote any position in the hallway, at which there is a door to the left of the robot (these are positions 1 and 4 in our example). The other three types are given by l_{DR},l_{WL} and l_{WR}. For example, l_{WR} are positions 0, 1 and 3. Now we can summarize the robot's ability to detect doors and walls as follows:

$$
\begin{aligned}
P(DL|L = l_{DL}) &= 0.7 \\
P(DL|L = l_{WL}) &= 0.1 \\
P(WL|L = l_{WL}) &= 0.9 \\
P(WL|L = l_{DL}) &= 0.3 \\
P(DR|L = l_{DR}) &= 0.7 \\
P(DR|L = l_{WR}) &= 0.1 \\
P(WR|L = l_{WR}) &= 0.9 \\
P(WR|L = l_{DR}) &= 0.3
\end{aligned}
$$

For example, the probability that the robot detects a door to its left if it is at location 3, is determined by $P(DL|L = l_{WL})$, since there is actually a wall to the robot's left side at position 3. Now we can start to estimate the position of the robot.

(a) Assume that the robot does not know where it is ($P(L)$ is given by equation (1)). The robot detects a wall to it's right, *i.e.* it makes the observation WR. What is the distribution of the robot's position estimate after this observation? Use a calculator (or write a computer program) and Bayes' rule.

(b) Assume the robot has already updated its position estimate based on the first observation WR. Now the robot turns the camera and looks to the left. What is the robot's position estimate, if it detects a door to the left? Use the distribution you computed in the previous step to represent $P(L = l_i)$. This probability distribution combines the fact that the robot has detected a wall to its right and a door to its left.

(c) After these two observations, the robot points the camera back to the right and, oops, it detects a door! Compute the new probability distribution and briefly discuss whether the new values make sense.

5. Section 8.1, Exercise 6, Parts a,c,e,g. (5th edition: Section 7.1, Exercise 6, Parts a,c,e,g.)

6. Draw the directed graph representing the following relation R on $\{0, 1, 2, \ldots, 10\}$: xRy if and only if $(x - y) \equiv 1 \pmod{11}$ or $xy \equiv 1 \pmod{11}$.

7. Let R be a random relation on the set $A = \{a_1, a_2, \ldots, a_n\}$ selected as follows: Independently for each pair i, j, $1 \le i \le n$ and $1 \le j \le n$, include (a_i, a_j) in R with probability $1/2$. Now,

   (a) What is the expected size of R?

   (b) What is the probability that R is reflexive?

   (c) What is the probability that R is symmetric?

   (d) What is the probability that R is anti-symmetric?