Problems:

1. Prove that if \( n \) is an integer, then \( n^2 \mod 8 \) is either 0, 1, or 4.

2. Compute the greatest common divisor for each of the following pairs of numbers.
   (a) \( 2^2 \cdot 3^3 \cdot 5^5, 2^5 \cdot 3^3 \cdot 5^4 \)
   (b) 1000, 625
   (c) 20!, 127

3. Prove that if \( a, b, \) and \( m \) are integers such that \( m \geq 2 \) and \( a \equiv b \pmod{m} \) then \( \gcd(a, m) = \gcd(b, m) \).

4. Use induction to show that \( 1^3 + 2^3 + \cdots + n^3 = \left\lfloor \frac{n(n+1)/2} \right\rfloor^2 \) whenever \( n \) is a positive integer.

5. Casting out nines: Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9. More specifically, for a positive integer \( a \), let \( \text{digitsum}(a) \) be the sum of the decimal digits of \( a \). Prove by induction on the number of decimal digits in \( a \) that \( a \equiv \text{digitsum}(a) \pmod{9} \).
   \textbf{Hint:} Express \( a \) as \( \sum_{i=0}^{n} a_i(10)^i \) where \( a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \).

6. Sometimes it’s easier to prove a stronger statement than is apparently required. In this problem you will prove by induction that for all \( n \geq 1 \),
   \[
   \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{n^3} \leq \frac{3}{2}.
   \]
   (a) What doesn’t work if you try to produce an inductive proof in which \( P(n) \) is the statement that \( \sum_{k=1}^{n} \frac{1}{k^3} \leq \frac{3}{2} \)?
   (b) Now use induction to prove the stronger statement that for all \( n \geq 1 \),
   \[
   \sum_{k=1}^{n} \frac{1}{k^3} \leq \frac{3}{2} - \frac{1}{2n^2}.
   \]

7. Find the flaw with the following “proof” that \( a^n = 1 \) for all non-negative integers \( n \), whenever \( a > 0 \).
   (a) \textbf{Basis step:} \( a^0 = 1 \) is true by the definition of \( a^0 \).
   (b) \textbf{Inductive step:}
   \[
   a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.
   \]

\textbf{EXTRA CREDIT ON BACK}
8. **Extra credit:** Use the results of problem 1 above to show that the equation

\[3x^2 - 2y^2 = 69\]

does not have any solution with \(x\) and \(y\) both integers.

9. **Extra credit:** Consider any \(n + 1\) numbers between 1 and \(2n\) (inclusive). Show that some pair of them is relatively prime. Show that one is a factor of another.