♦ Let $A$ and $B$ be sets. A **binary relation from $A$ to $B$** is a subset of $A \times B$. If $(a, b) \in R$, we write $aRb$ and say $a$ is related to $b$ by $R$.

♦ A **relation on** the set $A$ is a relation from $A$ to $A$.

♦ A relation $R$ on a set $A$ is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

♦ A relation $R$ on a set $A$ is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for $a, b \in A$.

♦ A relation $R$ on a set $A$ such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for $a, b \in A$, is called **antisymmetric**.

♦ A relation $R$ on a set $A$ is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for $a, b \in A$.
Combining Relations

◊ Let $R$ be a relation from a set $A$ to a set $B$ and $S$ be a relation from $B$ to a set $C$. The **composite** of $R$ and $S$ is the relation consisting of ordered pairs $(a, c)$, where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.
We denote the composite of $R$ and $S$ by $S \circ R$.

◊ Let $R$ be a relation on the set $A$. The **powers** $R^n$, $n = 1, 2, 3, \ldots$, are defined inductively by

\[
R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R.
\]

◊ **Theorem** : The relation $R$ on a set $A$ is transitive if and only if

\[
R^n \subseteq R \text{ for } n = 1, 2, 3, \ldots.
\]
Let $P$ be a property of relations (transitivity, reflexivity, symmetry). A relation $S$ is closure of $R$ w.r.t. $P$ if and only if $S$ has property $P$, $S$ contains $R$, and $S$ is a subset of every relation with property $P$ containing $R$. 
A directed graph, or digraph, consists of a set $V$ of vertices (or nodes) together with a set $E$ of ordered pairs of elements of $V$ called edges (or arcs).

A path from $a$ to $b$ in the directed graph $G$ is a sequence of one or more edges $(x_0, x_1), (x_1, x_2), \ldots (x_{n-1}, x_n)$ in $G$, where $x_0 = a$ and $x_n = b$. This path is denoted by $x_0, x_1, \ldots, x_n$ and has length $n$. A path that begins and ends at the same vertex is called a circuit or cycle.

There is a path from $a$ to $b$ in a relation $R$ is there is a sequence of elements $a, x_1, x_2, \ldots x_{n-1}, b$ with $(a, x_1) \in R, (x_1, x_2) \in R, \ldots, (x_{n-1}, b) \in R$.

Theorem: Let $R$ be a relation on a set $A$. There is a path of length $n$ from $a$ to $b$ if and only if $(a, b) \in R^n$. 
Connectivity

♦ Let $R$ be a relation on a set $A$. The **connectivity relation** $R^*$ consists of pairs $(a, b)$ such that there is a path between $a$ and $b$ in $R$.

♦ **Theorem:** The transitive closure of a relation $R$ equals the connectivity relation $R^*$. 

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Partitions

◊ We want to use relations to form partitions of a group of students. Each member of a subgroup is related to all other members of the subgroup, but to none of the members of the other subgroups.

◊ Use the following relations:
  
  Partition by the relation "older than"
  Partition by the relation "partners on some project with"
  Partition by the relation "comes from same hometown as"

◊ Which of the groups will succeed in forming a partition? Why?
A relation on a set $A$ is called an equivalence relation if it is reflexive, symmetric, and transitive. Two elements that are related by an equivalence relation are called equivalent.

Let $R$ be an equivalence relation on a set $A$. The set of all elements that are related to an element $a$ of $A$ is called the equivalence class of $a$. $[a]_R$: equivalence class of $a$ w.r.t. $R$. If $b \in [a]_R$ then $b$ is representative of this equivalence class.

**Theorem:** Let $R$ be an equivalence relation on a set $A$. The following statements are equivalent:

1. $aRb$
2. $[a] = [b]$
3. $[a] \cap [b] \neq \emptyset$
A partition of a set $S$ is a collection of disjoint nonempty subsets $A_i, i \in I$ (where $I$ is an index set) of $S$ that have $S$ as their union:

- $A_i \neq \emptyset$ for $i \in I$
- $A_i \cap A_j = \emptyset$, when $i \neq j$
- $\bigcup_{i \in I} A_i = S$

Theorem: Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i, i \in I$, as its equivalence classes.