CSE 321: Discrete Structures Spring 2004

Problem Set 6

Due Friday, May 21, 2004, in class

Note: There are **EIGHT** problems, including an optional *Bonus* problem.

All exercise numbers refer to the number in Rosen's book, 5th Edition.

1. Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.
2. Section 5.2, Exercise 28.
3. (a) What is the conditional probability that exactly three heads appear when a fair coin is flipped five times, given that the first flip came up heads?
   (b) Suppose we choose randomly and independently two subsets A and B from the set of all possible non-empty subsets of $\{1, 2, \ldots, n\}$. What is the probability that $\min(A) = \min(B)$ (where $\min(A)$ denotes the minimum number from the set A).
4. Section 5.1, Exercise 38.
5. Suppose the UW Huskies play a best-of-5 football tournament with the Cal Bears (so the first team to achieve 3 victories wins the tournament). The probability for each team to win the first game is 1/2. However, a team that wins a game (not necessarily the first one) gets a morale boost, and so its probability to win after a victory increases to 2/3. Similarly, after a loss, the probability of a win decreases to 1/3.
   (a) What is the probability that the UW wins the tournament?
   (b) Given that UW loses the first two games to Cal, what is the probability that UW makes a striking comeback to win the tournament?
6. Great Expectations:
   (a) Suppose a 6-sided fair dice is rolled. Let the random variable X be the value showing. What is the expectation of X? Suppose two fair 6-sided dice are rolled independently. Let Y be the random variable which is the sum of the two values showing. What is the expectation of Y? Let Z be the random variable which is the minimum of the two values showing. What is the expected value of Z?
   (b) Suppose that a fair coin is tossed 1000 times. Let X be the random variable which is the number of flips i in which the coin takes the same value in both flip i and $i+1$. What is the expected value of X? (Just to clarify, in the sequence $HHHH$, X is 3, and in the sequence $THHHTT$, X is also 3, etc.)
7. Suppose you play the following game. Your rival has 100 blank cards, and on each of them she writes an arbitrary integer at will (and never repeats an integer on two cards). Then the cards are perfectly shuffled, and the shuffled deck is laid face down on a table so that the numbers are not visible. You start removing cards from the top one by one and look at their numbers. After turning any of the cards, you can end the game. You win the game if and

only if the last card you turned has the largest number among all the cards (those already turned but also those still lying on the table).

The rival proposes the following stakes to you: if you win she will give you 20 dollars, and if you lose you must pay 5 dollars. Given your wisdom from taking 321, will you play this game? That is, is there a strategy the expectation of whose return is positive (so it is worth your while using that strategy to play the game)? Why, or why not? (If you claim there is no such strategy, prove your answer, and if you claim there is such a positive yield strategy, give one!)

8. * **(Bonus Problem)** Let n be an odd number of women. Let each have a uniformly chosen random bit on her forehead. Each person can see all the bits except her own. They wish to vote on the parity of the bits. No communication between the voters is allowed. Formally, each person casts a private 0 or 1 vote; the outcome of the election is the value which the majority cast. The n women are said to win the election in the case when the outcome is equal to the parity of the n bits.

   It might be tempting to believe that the voters as a group will be able to win only half the time, since no individual voter ever learns any information about the parity of the bit and thus will be wrong exactly half of the time. Yet, your task is to come up with *a collective strategy for the voters which gives them a high probability, specifically a probability that tends to* 1 *as* $n \to \infty$*, of winning the election!*