Discrete Probability

◊ **Probability**: The probability of an event $E$, which is a subset of a finite sample space $S$ of equally likely outcomes, is

$$p(E) = \frac{|E|}{|S|}.$$  

◊ **Theorem**: Let $E$ be an event in a sample space $S$. The probability of the event $\bar{E}$, the complementary event of $E$, is given by

$$p(\bar{E}) = 1 - p(E).$$

◊ **Theorem**: Let $E_1$ and $E_2$ be events in a sample space $S$. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$
Let $S$ be the sample space of an experiment with a finite or countable number of outcomes. We assign probability $p(s)$ to each outcome $s$. The following two conditions have to be met:

(i) $0 \leq p(s) \leq 1$ for each $s \in S$

(ii) $\sum_{s \in S} p(s) = 1$

The probability of the event $E$ is the sum of the probabilities of the outcomes in $E$. That is,

$$p(E) = \sum_{s \in E} p(s).$$
Conditional Probability

Let $E$ and $F$ be events with $p(F) > 0$. The \textbf{conditional probability} of $E$ given $F$ is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

The events $E$ and $F$ are said to be \textbf{independent} if and only if

$$p(E \cap F) = p(E)p(F).$$
Bernoulli Trial

◊ **Bernoulli Trial**: Experiment with only two possible outcomes: success or failure.

◊ **Probability of $k$ successes in $n$ independent Bernoulli trials**
  with probability of success $p$ and probability of failure $q = 1 - p$, is

\[
\binom{n}{k} p^k q^{n-k}.
\]
Random Variables

◊ A **random variable** is a function from the sample space of an experiment to the set of real numbers. That is a random variable assigns a real number to each possible outcome.

◊ The **distribution** of a random variable $X$ on a sample space $S$ is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$, where $p(X = r)$ is the probability that $X$ takes the value $r$. A distribution is usually described by specifying $p(X = r)$ for each $r \in X(S)$. 
The expected value (or expectation) of a random variable $X(s)$ on the sample space $S$ is equal to

$$E(X) = \sum_{s \in S} p(s)X(s).$$

**Theorem**: If $X$ and $Y$ are random variables on a space $S$, then

$$E(X + Y) = E(X) + E(Y).$$

Furthermore, if $X_i, i = 1, 2, \ldots, n$, with $n$ a positive integer, are random variables on $S$, and $X = X_1 + X_2 + \ldots + X_n$, then

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_n).$$

Moreover, if $a$ and $b$ are real numbers, then $E(aX + b) = aE(X) + b$. 
The random variables $X$ and $Y$ on a sample space $S$ are independent if for all real numbers $r_1$ and $r_2$

$$p(X(s) = r_1 \text{ and } Y(s) = r_2) = p(X(s) = r_1) \cdot p(Y(s) = r_2).$$

**Theorem**: If $X$ and $Y$ are independent random variables on a space $S$, then $E(XY) = E(X)E(Y)$. 

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Variance

◊ Let $X$ be random variables on a sample space $S$. The variance of $X$, denoted by $V(X)$, is

$$V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s).$$

The standard deviation of $X$, denoted $\sigma(X)$, is defined to be $\sqrt{V(X)}$.

◊ Theorem: If $X$ is a random variable on a space $S$, then

$$V(X) = E(X^2) - E(X)^2.$$

◊ Theorem: If $X$ and $Y$ are two independent random variables on a space $S$, then $V(X + Y) = V(X) + V(Y)$. Furthermore, if $X_i, i = 1, 2, \ldots, n$ with $n$ a positive integer, are pairwise random variables on $S$, and

$$X = X_1 + X_2 + \ldots + X_n,$$

then $V(X_1 + X_2 + \ldots + X_n) = V(X_1) + V(X_2) + \ldots + V(X_n)$. 