1. Prove that if \( n \) is an odd positive integer, then \( n^4 \equiv 1 \pmod{16} \).

2. A **perfect number** is a positive integer that equals the sum of its proper divisors (that is, devisors other than itself). Show that 6, 28, and 496 are perfect.

3. Prove that \( 1^2 + 3^2 + 5^2 + \ldots + (2n + 1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3} \) whenever \( n \) is a nonnegative integer.

4. Show that \( 2^n > n^2 \) whenever \( n \) is a positive integer greater than 4.

5. An automatic teller machine has only $20 bills and $50 bills. Which amount of money can the machine dispense, assuming the machine has a limitless supply of there two denominations of bills? Prove your answer using a form of mathematical induction.

6. A multiple-choice test contains ten questions. There are four possible answers for each question.

   a. How many ways can a student answer the questions on the test if every question is answered?
   
   b. How many questions can a student answer the questions on the test if the student can leave answers blank?
1. \[ n^4 - 1 = (2k + 1)^4 - 1 = 16k^4 + 32k^3 + 24k^2 + 8k = 8k(1)(2k^2 + 2k + 1) \] One of \( k \) or \( k+1 \) is even, so 16 divides \( n^4 - 1 \).

2. \[
\begin{align*}
1+2+3 &= 6 \\
1+2+4+7+14 &= 28 \\
1+2+4+8+16+31+62+124+248 &= 496
\end{align*}
\]

3. Let \( P(n) \) be \( \text{“} \, 1^2 + 3^2 + \ldots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3. \text{”} \)

   **Basis Step:** \( P(0) \) is true since \( 1^2 = 1 = (0+1)(2*0+1)(2*0+3)/3 \)

   **Inductive Step:** Assume that \( P(k) \) is true. Then

   \[
   1^2 + 3^2 + \ldots + (2k+1)^2 + (2(k+1) + 1)^2 = \frac{(k + 1)(2k + 1)(2k + 3)}{3} + (2k + 3)^2
   \]

   \[
   = (2k + 3)\left[\frac{(k + 1)(2k + 1)}{3} + (2k + 3)\right] = \frac{(2k + 3)(2k^2 + 9k + 10)}{3}
   \]

   \[
   = \frac{(2k + 3)(2k + 5)(k + 2)}{3} = \frac{(k + 1)(k + 1)(2(k + 1) + 1)(2(k + 1) + 3)}{3}
   \]

4. Let \( P(n) \) be \( \text{“} \, 2^n > n^2. \text{”} \)

   **Basis Step:** \( P(5) \) is true since \( 2^5 = 32 > 25 = 5^2 \).

   **Inductive Step:** Assume that \( P(k) \) is true, that is, \( 2^k > k^2 \). Then

   \[ 2^{k+1} = 2 \cdot 2^k > k^2 + k^2 > k^2 + 4k + 2k + 1 = (k + 1)^2 \]

   since \( k > 4 \).

5. All multiples of $10$ greater than or equal to $40$ can be formed as well as $20$. Let \( P(n) \) be the statement that $10n$ dollars can be formed. \( P(4) \) is true since $40$ can be formed by using two $20$s. Now assume that \( P(k) \) is true with \( k > 4 \). If a $50$ bill is used to form $10k$ dollars, replace it by three $20$ bills to obtain $10(k+1)$ dollars. Otherwise, at least two $20$ bills were used since $10k$ is at least $40$. Replace two $20$ bills with a $50$ bill to obtain $10(k+1)$. This shows that \( P(k+1) \) is true.

6. a) \( 4^{10} \)
   b) \( 5^{10} \)