Undirected Graphs

◊ A **simple graph** \( G = (V, E) \) consists of \( V \), a nonempty set of vertices, and \( E \), a set of unordered pairs of distinct elements of \( V \) called edges.

◊ A **multigraph** \( G = (V, E) \) consists of a set \( V \) of vertices, a set \( E \) of edges, and a function \( f \) from \( E \) to \( \{ \{u, v\} \mid u, v \in V, u \neq v \} \). The edges \( e_1 \) and \( e_2 \) are called **multiple** or **parallel edges** if \( f(e_1) = f(e_2) \).

◊ A **pseudograph** \( G = (V, E) \) consists of a set \( V \) of vertices, a set \( E \) of edges, and a function \( f \) from \( E \) to \( \{ \{u, v\} \mid u, v \in V \} \). An edge is a **loop** if \( f(e) = \{u, u\} = \{u\} \) for some \( u \in V \).
A directed graph $G = (V, E)$ consists of a set $V$ of vertices and a set of edges $E$ that are ordered pairs of elements of $V$.

A directed multigraph $G = (V, E)$ consists of a set $V$ of vertices, a set $E$ of edges, and a function $f$ from $E$ to $\{(u, v) \mid u, v \in V\}$. The edges $e_1$ and $e_2$ are multiple edges if $f(e_1) = f(e_2)$. 
Undirected Graph Terminology

♦ Two vertices $u$ and $v$ in an undirected graph $G$ are called adjacent (or neighbors) in $G$ if $\{u, v\}$ is an edge of $G$. If $e = \{u, v\}$, the edge $e$ is called incident with the vertices $u$ and $v$. The edge $e$ is also said to connect $u$ and $v$. The vertices $u$ and $v$ are called endpoints of the edges $\{u, v\}$.

♦ The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex $v$ is denoted by $\text{deg}(v)$.

♦ The Handshaking Theorem: Let $G = (V, E)$ be an undirected graph with $e$ edges. Then

\[
2e = \sum_{v \in V} \text{deg}(v).
\]

♦ Theorem: An undirected graph has an even number of vertices of odd degree.
Directed Graph Terminology

◊ When \((u, v)\) is an edge of the graph \(G\) with directed edges, \(u\) is said to be **adjacent to** \(v\) and \(v\) is said to be **adjacent from** \(u\). The vertex \(u\) is called the **initial vertex** of \((u, v)\), and \(v\) is called the **terminal** or **end vertex** of \((u, v)\). The initial vertex and terminal vertex of a loop are the same.

◊ In a graph with directed edges the **in-degree** of a vertex \(v\), denoted by \(\text{deg}^- (v)\), is the number of edges with \(v\) as their terminal vertex. The **out-degree** of \(v\), denoted by \(\text{deg}^+ (v)\), is the number of edges with \(v\) as their initial vertex.

◊ **Theorem:** Let \(G = (V, E)\) be a graph with directed edges. Then

\[
\sum_{v \in V} \text{deg}^- (v) = \sum_{v \in V} \text{deg}^+ (v) = |E|.
\]
More Definitions . . .

◊ A simple graph is \( G \) is called **bipartite** if its vertex \( V \) can be partitioned into two disjoint nonempty sets \( V_1 \) and \( V_2 \) such that every edge in the graph connects a vertex in \( V_1 \) and a vertex in \( V_2 \) (so that no edge in \( G \) connects either two vertices in \( V_1 \) or two vertices in \( V_2 \).

◊ A **subgraph** of a graph \( G = (V, E) \) is a graph \( H = (W, F) \) where \( W \subseteq V \) and \( F \subseteq E \).

◊ The **union** of two simple graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) is the simple graph with vertex set \( V_1 \cup V_2 \) and edge set \( E_1 \cup E_2 \). The union of \( G_1 \) and \( G_2 \) is denoted by \( G_1 \cup G_2 \).

◊ The simple graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are **isomorphic** if there is a one-to-one and onto function \( f \) from \( V_1 \) to \( V_2 \) with the property that \( a \) and \( b \) are adjacent in \( G_1 \) if and only if \( f(a) \) and \( f(b) \) are adjacent in \( G_2 \), for all \( a \) and \( b \) in \( V_1 \). Such a function \( f \) is called an **isomorphism**.