CSE 312

Foundations of Computing II

Lecture 9: Linearity of expectation, LOTUS and variance

slido.com/3680281

Agenda

- Recap
- Linearity of expectation
- LOTUS
- Variance

Review Random Variables

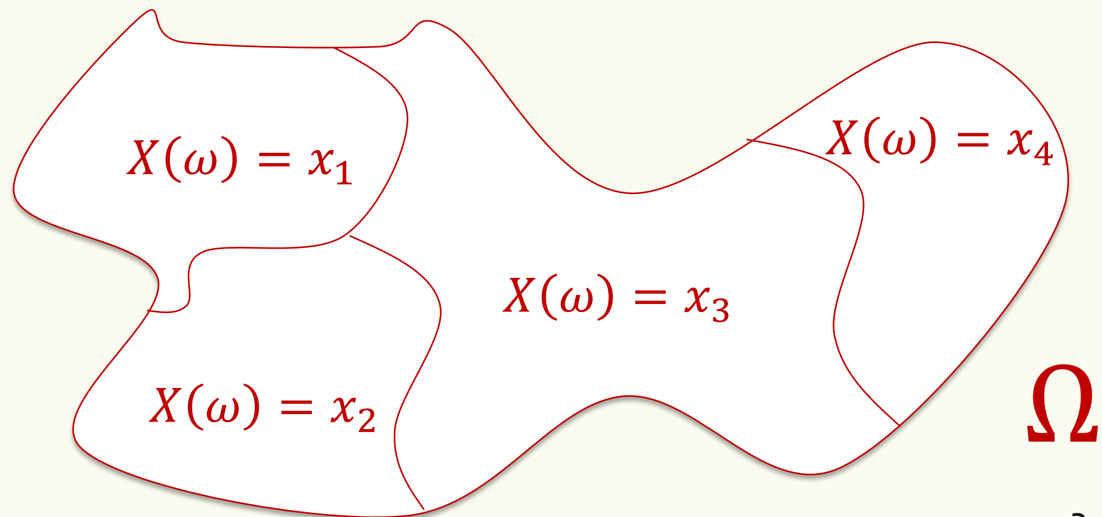
Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is its *range/support*: $\cancel{X(\Omega)}$ Ω_X

$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables **partition** the sample space.

$\Sigma_{x \in \cancel{X(\Omega)}\,\Omega_X} P(X = x) = 1$



Review PMF and CDF

Definitions:

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of X specifies, for any real number x, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\boxed{\sum_{x \in \Omega_X} p_X(x) = 1}$$

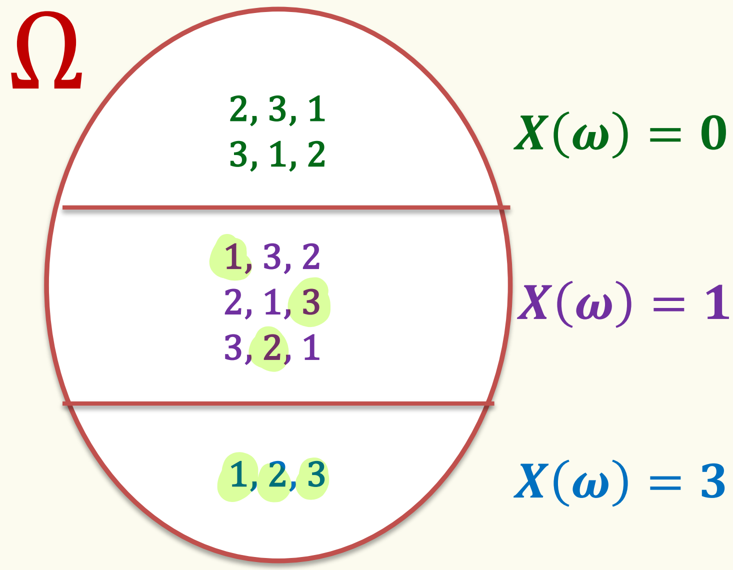
For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function (cdf) of X specifies, for any real number x, the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$

Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1



Review Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\boxed{\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)} = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

X takes value 5 with prob $\frac{1}{5}$

10 with prob $\frac{4}{5}$

$$E(X) = 5 \cdot \frac{1}{5} + 10 \cdot \frac{4}{5} = 9$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Example: Returning Homeworks

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x)$$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$
$1/6$	$1, 2, 3$	3
$1/6$	$1, 3, 2$	1
$1/6$	$2, 1, 3$	1
$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1

$$\mathbb{E}[X] = 3 \cdot \overset{\frac{1}{6}}{P(X = 3)} + 1 \cdot \overset{\frac{1}{2}}{P(X = 1)} + 0 \cdot \overset{\frac{1}{3}}{P(X = 0)}$$

$$= P(123) \quad P(132) + P(213) + P(321) \quad P(231) + P(312)$$

$$= \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

Review Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \quad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Indicator random variable – 0/1 valued

- Class with 3 students, randomly hand back homeworks.
  All permutations equally likely.
- For any event, can define the indicator random variable for that event

$$X_1 = \begin{cases} 1 & \text{if person 1 gets their homework back} \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$	$X_1(\omega)$
1/6	1, 2, 3	3	1
1/6	1, 3, 2	1	1
1/6	2, 1, 3	1	0
1/6	2, 3, 1	0	0
1/6	3, 1, 2	0	0
1/6	3, 2, 1	1	0

$P(X_1 = 1) = \frac{1}{3}$

$P(X_1 = 0) = \frac{2}{3}$

$$E(X_1) = 1 \cdot P(X_1 = 1) + 0 \cdot P(X_1 = 0)$$

$$= P(X_1 = 1) = \frac{1}{3}$$

Recap Linearity of Expectation

$X(\omega) + Y(\omega)$

Theorem. For any two random variables X and Y

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

$E(a_1 X_1 + \cdots + a_n X_n + b) = a_1 E(X_1) + \cdots + a_n E(X_n) + b$

Theorem. For any random variables X, and constants a and b

$$\mathbb{E}[aX + b] = a \cdot \mathbb{E}[X] + b.$$

$Y = 3X - 5$

$E(Y) = 3E(X) - 5$

Example – Coin Tosses – The brute force method

We flip n coins, each one heads with probability p,

Z is the number of heads, what is $\mathbb{E}[Z]$?

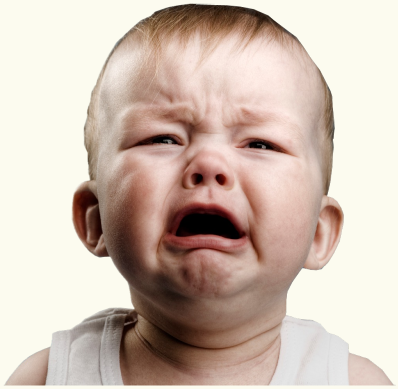
$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) \quad = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$



Can we solve it more elegantly, please?

Example – Coin Tosses

We flip n coins, each toss independent, comes up heads with probability p

Z is the number of heads, what is $\mathbb{E}[Z]$?

$$X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$$

Fact. $Z = X_1 + \cdots + X_n$

Outcomes	X_1	X_2	X_3	Z
TTT	0	0	0	0
TTH	0	0	1	1
THT	0	1	0	1
THH	0	1	1	2
HTT	1	0	0	1
HTH	**1**	**0**	**1**	**2**
HHT	1	1	0	2
HHH	1	1	1	3

$E(Z) = E(X_1 + X_2 + X_3)$

$= E(X_1) + E(X_2) + E(X_3)$

$= p + p + p = 3p$

$E(X_i) = 1 \cdot P(X_i = 1)$

$+ 0 \cdot P(X_i = 0)$

$= P(X_i = 1)$

$= p$

Example – Coin Tosses

We flip n coins, each toss independent, comes up heads with probability p

Z is the number of heads, what is $\mathbb{E}[Z]$?

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$

Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \underbrace{\mathbb{E}[X_1]}_{p} + \cdots + \underbrace{\mathbb{E}[X_n]}_{p} = n \cdot p$$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

$$\mathbb{E}[X_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$

linearity of expectation

Using LOE to compute complicated expectations

$E(X) = ?$

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

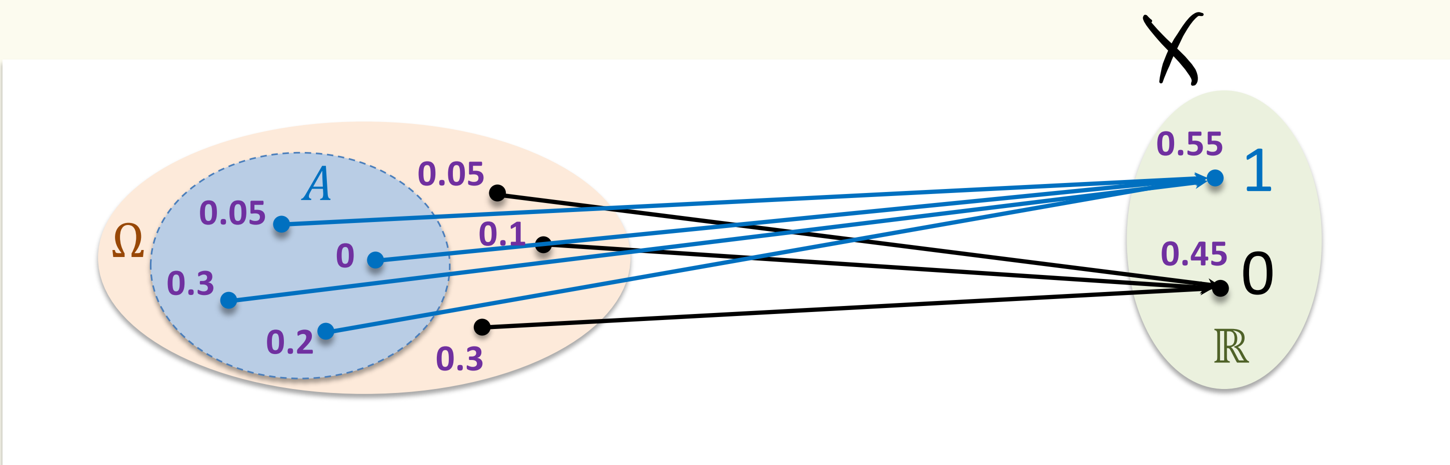
Often, X_i are **indicator** (0/1) random variables.

Indicator random variables – 0/1 valued

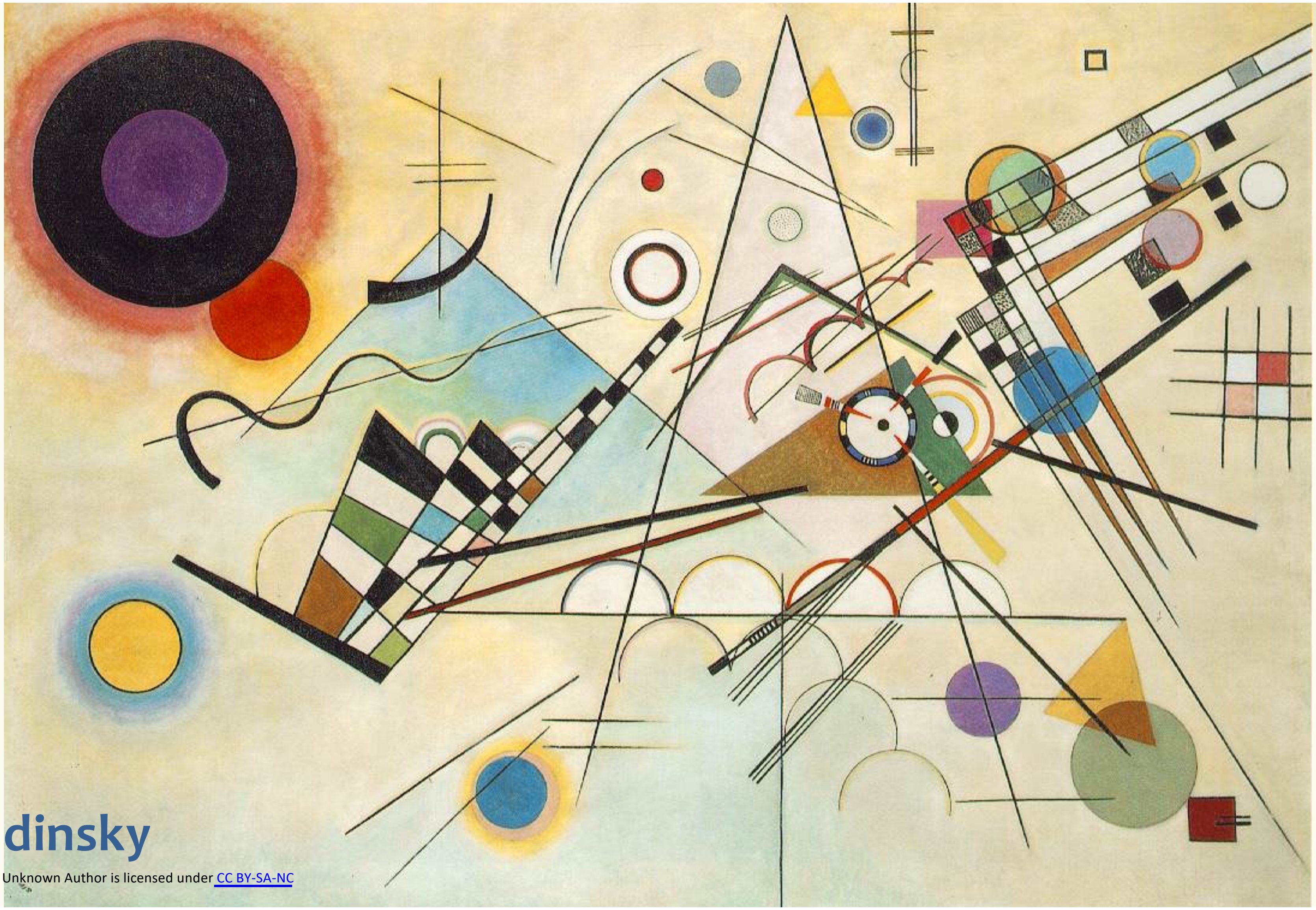
For any event A, can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$\boxed{\begin{array}{l} P(X_A = 1) = P(A) \\ P(X_A = 0) = 1 - P(A) \end{array}}$$



$$E(X_A) = P(X_A = 1) = P(A)$$



Kandinsky

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$?

$$E(X) = \sum_{k=0}^{n} k \underbrace{P(X=k)}_{\text{complicated!}}$$

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1/6	1, 2, 3	3
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Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
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What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
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$1/6$	$3,1,2$	0
$1/6$	$3,2,1$	1

Decompose: Find the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Conquer: Compute the expectation of each X_i and sum!

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
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1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Decompose: $X_i = \begin{cases} 1 & \text{if student } i \text{ get their own hw back} \\ 0 & \text{o.w.} \end{cases}$

$X = X_1 + X_2 + \dots + X_n$

LOE: $E(X) = \underbrace{E(X_1)}_{\frac{1}{n}} + \underbrace{E(X_2)}_{\frac{1}{n}} + \dots + \underbrace{E(X_n)}_{\frac{1}{n}}$

Conquer: $= n \cdot \frac{1}{n} = 1$

$E(X_i) = \Pr(\text{student } i \text{ get their own hw back}) = \frac{1}{n}$

$\frac{(n-1)!}{n!} = \frac{1}{n}$

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\Pr(\omega)$	ω	$X(\omega)$
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$1/6$	$2,3,1$	0
$1/6$	$3,1,2$	0
$1/6$	$3,2,1$	1

Decompose: What is X_i?

$X_i = 1$ iff i^{th} student gets own HW back; 0 o.w.

LOE: $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: $\mathbb{E}[X_i] = \frac{1}{n}$

Therefore, $\mathbb{E}[X] = n \cdot \frac{1}{n} = 1$

#poss 365^m

X # pairs that have same bday

$m=6$ $X(\omega)=4$

ω	1	2	3	4	5	6
	Aug 1	Oct 21	Oct 21	Oct 21	Nov 1	Nov 1

$\boxed{X} = X_{12}+X_{13}+X_{14}+X_{15}+X_{16}+X_{23}+X_{24}+X_{25}+X_{26}+X_{34}+X_{35}+X_{36}+X_{45}+X_{46}+X_{56}$

Pairs with the same birthday

In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?
(Each person's birthday is equally likely to be any of the 365 possibilities and different people's bdays are independent.)

$E(X) = ?$ $= \sum_{k=0}^{\binom{m}{2}} k\, \underbrace{P(k \text{ pairs have same bday})}_{P(X=k)}$

$1, 2, \ldots, m$

$X_{ij} = \begin{cases} 1 & \text{if } i \,\&\, j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$

$1 \le i < j \le m$

$$\boxed{E(X) = \sum_{x \in \Omega_X} x\, P(X=x)}$$

$$E(X) = \sum_{1 \le i < j \le m} E(X_{ij}) = \binom{m}{2}\frac{1}{365}$$

$E(X_{ij}) = P(i \,\&\, j \text{ have same bday}) = \frac{1}{365}$

$$\sum_{b=1}^{365} P(i \text{ has } b \cap j \text{ has } b) = \sum_{b=1}^{365} \frac{1}{365} \cdot \frac{1}{365}$$

$$\underbrace{P(i \text{ has } b)}_{\frac{1}{365}} P(j \text{ has } b)$$

$$\cdot\, \cancel{365} \cdot \frac{1}{\cancel{365}} \cdot \frac{1}{365} = \frac{1}{365}$$

Pairs with the same birthday

- In a class of m students, on average how many pairs of people have the same birthday (assuming 365 equally likely birthdays)?

Decompose: Indicator events involve **pairs** of students (i, j) for $i \neq j$

$X_{ij} = 1$ iff students i and j have the same birthday

LOE: $\binom{m}{2}$ indicator variables X_{ij}

Conquer: $\mathbb{E}[X_{ij}] = \frac{1}{365}$ so total expectation is $\frac{\binom{m}{2}}{365} = \frac{m(m-1)}{730}$ pairs

Agenda

- Recap
- Linearity of expectation
- LOTUS
- Variance

Linearity of Expectation – Even stronger

Theorem. For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}[a_1 X_1 + \cdots + a_n X_n] = a_1 \mathbb{E}[X_1] + \cdots + a_n \mathbb{E}[X_n].$$

Very important: In general, we do <u>not</u> have $\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Linearity is special!

In general $\mathbb{E}[g(X)] \neq g\big(\mathbb{E}(X)\big)$

$\boxed{g(x)=x^2}$

$E(X)=0$

E.g., $X = \begin{cases} +1 \text{ with prob } 1/2 \\ -1 \text{ with prob } 1/2 \end{cases}$

x^2
1
1

$E(X^2)=1$

Then: $\underline{\mathbb{E}[X^2]} \neq \underline{\mathbb{E}[X]^2}$

How DO we compute $\mathbb{E}[g(X)]$?

Expected Value of $g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot P(\omega)$$

$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\omega)$$

or equivalently

$$E(X) = \sum_{x \in \Omega_X} x \, P(X = x)$$

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x) = \sum_{x \in \Omega_X} g(x) \cdot p_X(x)$$

Also known as **LOTUS**: "Law of the unconscious statistician

(nothing special going on in the discrete case)

Example: from concept check

$$\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) \cdot P(X = x)$$

- Toss a die; each side equally likely. X is the number showing
- $\boxed{Y = X \bmod 4}$
- What is $\mathbb{E}[Y]$?

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$\mathbf{Pr}(\boldsymbol{\omega})$	$\boldsymbol{\omega}$	$\boldsymbol{X}$
1/6	1	1
1/6	2	2
1/6	3	3
1/6	4	4
1/6	5	5
1/6	6	6