CSE 312

Foundations of Computing II

Lecture 8: More on random variables; expectation

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Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)

Today:

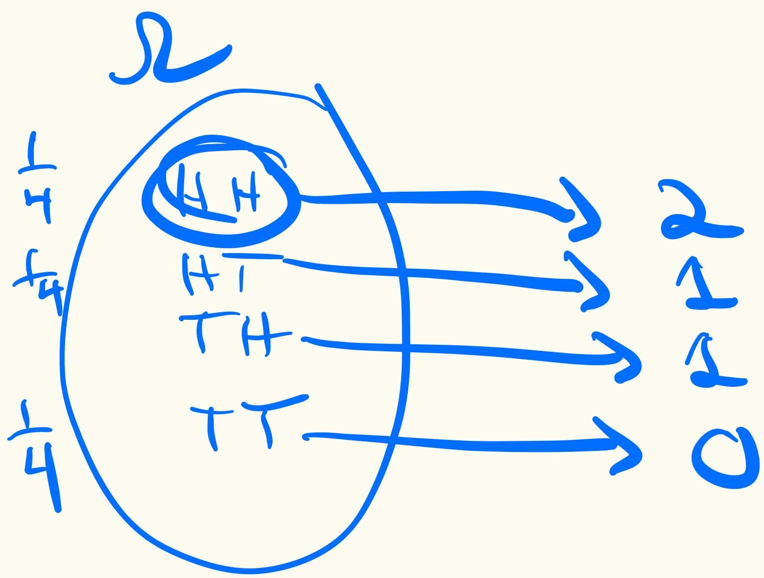
- Recap
- Expectation
- Linearity of Expectation
- Indicator Random Variables

Kandinsky

Review Random Variables

Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X : \Omega \to \mathbb{R}$.

The set of values that X can take on is its *range/support*: Ω_X



X: # heads

$\{0, 1, 2\}$

Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$\Omega_X = \{0, 1, 3\}$

Review Random Variables

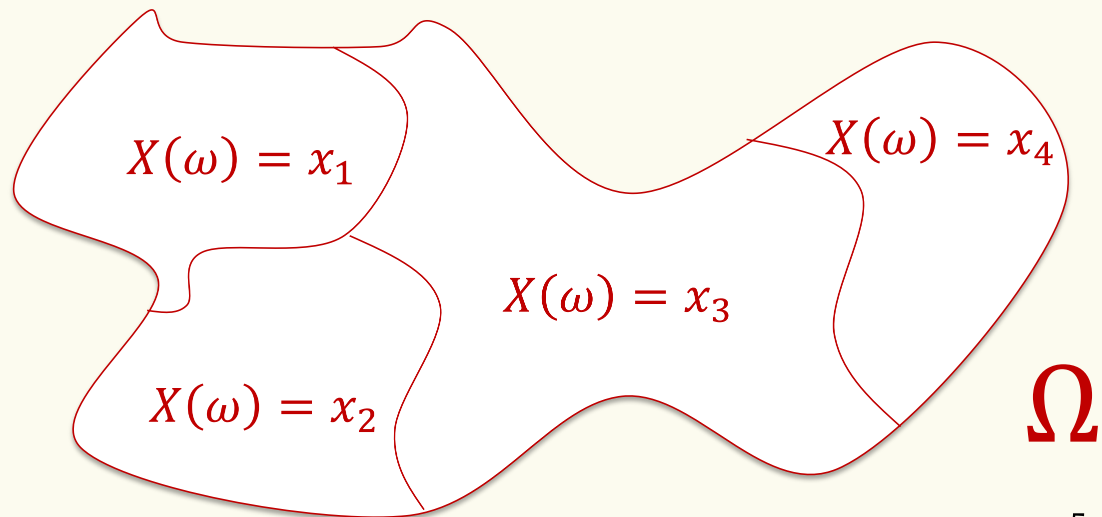
Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is its *range/support*: $\cancel{X(\Omega)}$ Ω_X

$$\boxed{\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}}$$

Random variables **partition** the sample space.

$$\Sigma_{x \in X(\Omega)} P(X = x) = 1$$



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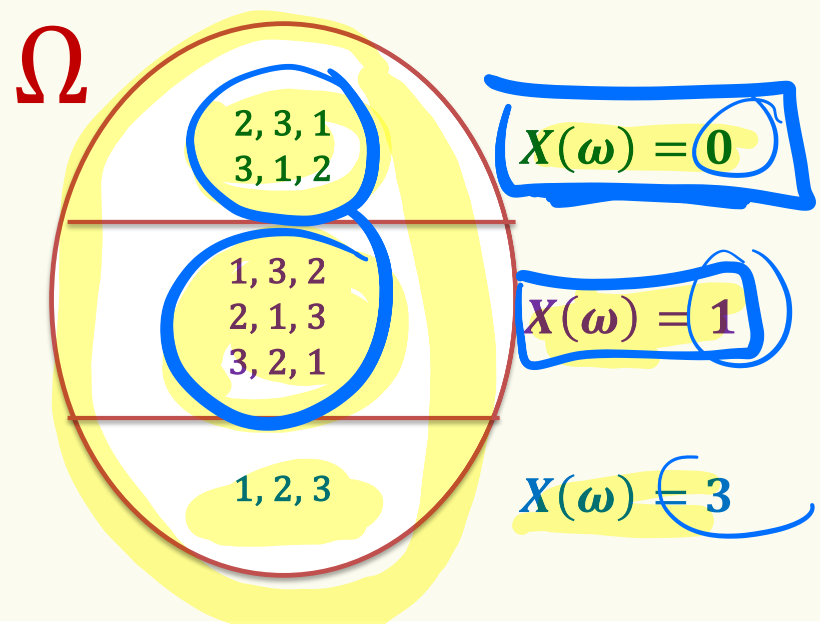


$$\sum_{x\in\Omega_X} P(X=x) = 1$$

Review PMF and CDF

Definitions:

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of X specifies, for any real number x, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\boxed{\sum_{x \in \Omega_X} p_X(x) = 1}$$

For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function (cdf) of X specifies, for any real number x, the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$

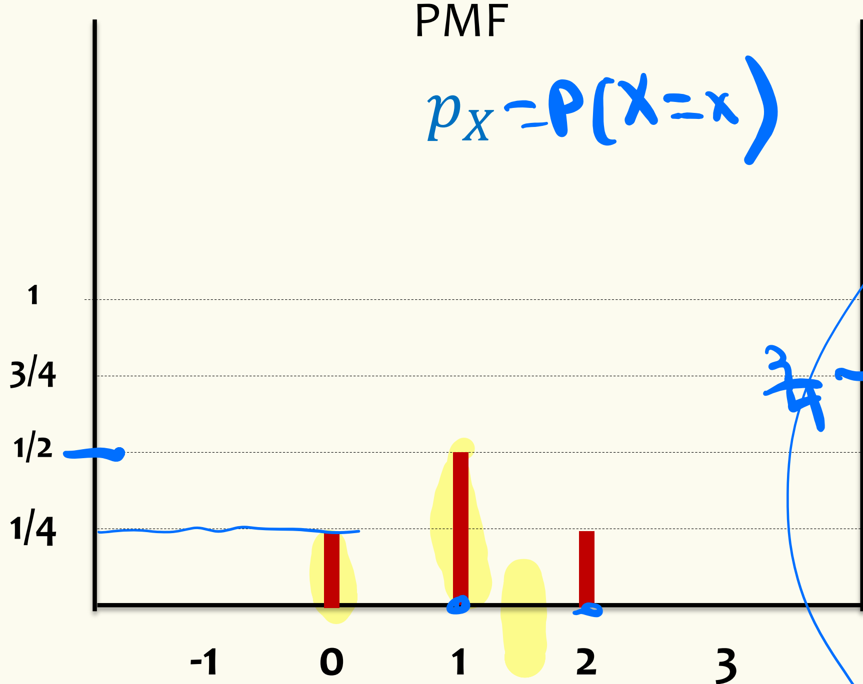
Example – Two fair independent coin flips

X = number of heads

$\Omega_X = \{0, 1, 2\}$

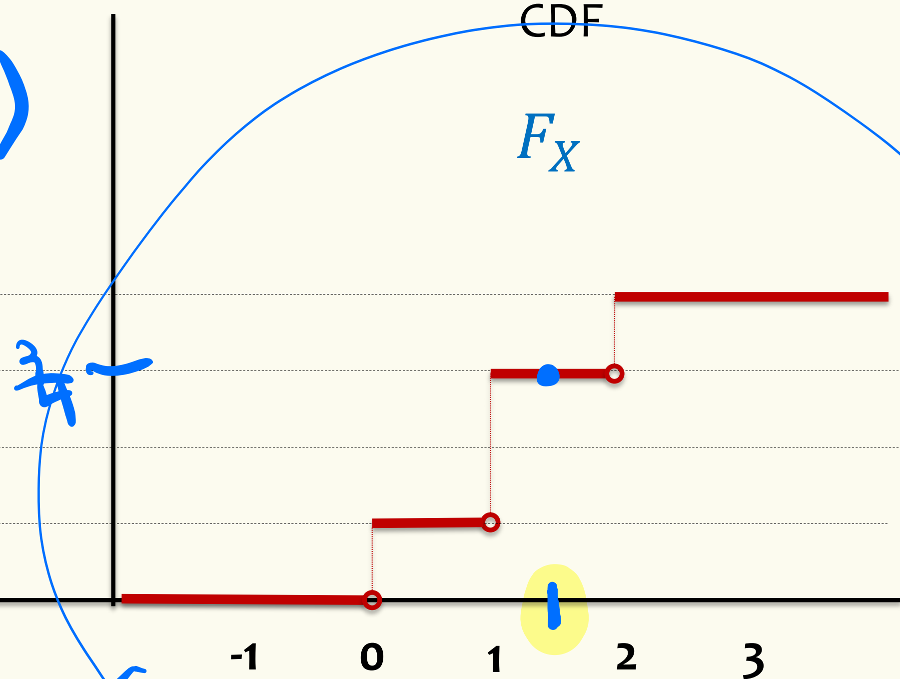
Probability Mass Function
PMF

$p_X = P(X = x)$



Cumulative Distribution Function
CDF

F_X



$P(X \le 1.5) = P(X=0) + P(X=1)$

Example: Returning Homeworks

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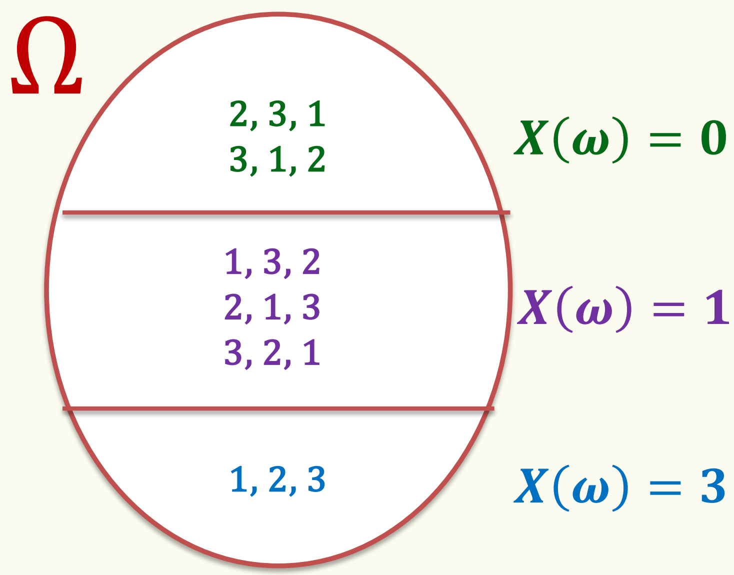
$1/6$	$1, 3, 2$	1
$1/6$	$2, 1, 3$	1
$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1



Example: Returning Homeworks

$\Pr(\omega)$	ω	$X(\omega)$
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1/6	3, 2, 1	1

- Class with 3 students, randomly hand back homeworks.

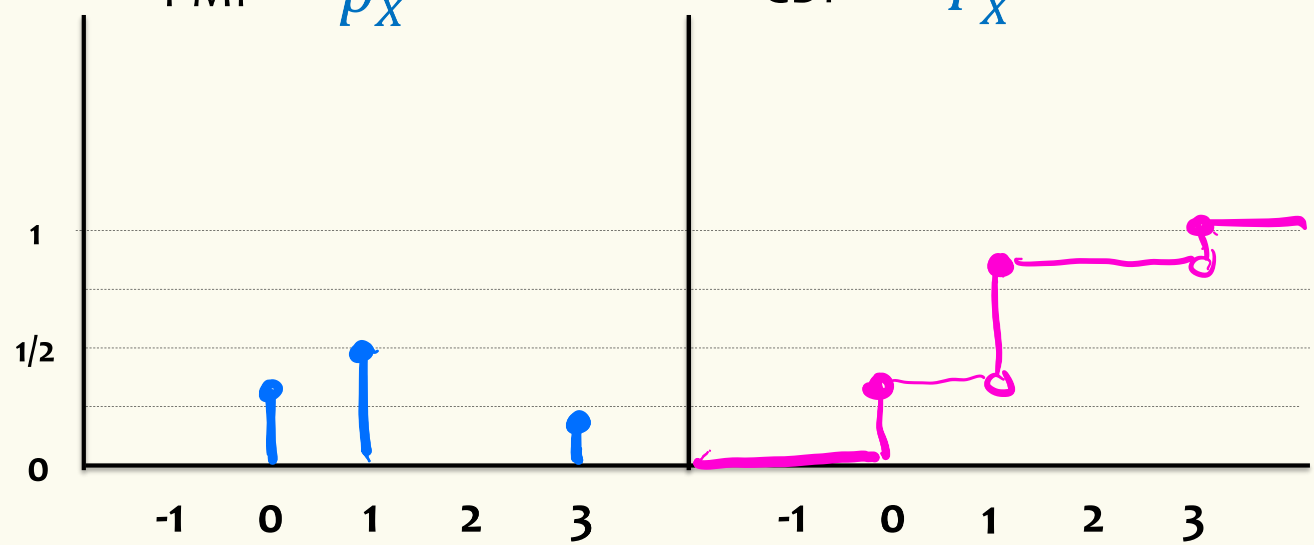
All permutations equally likely.

- **Let X be the number of students who get their own HW**

Probability Mass Fn
PMF p_X

Cumulative Distribution Fn
CDF F_X

$$\mathbf{p_X(0)} = P(X = 0) = 1/3$$
$$\mathbf{p_X(1)} = P(X = 1) = 1/2$$
$$\mathbf{p_X(3)} = P(X = 3) = 1/6$$



Example – Number of Heads

We flip n coins, independently, each heads with probability p

$$\Omega = \{\text{HH}\cdots\text{HH}, \text{HH}\cdots\text{HT}, \text{HH}\cdots\text{TH}, \ldots, \text{TT}\cdots\text{TT}\}$$

$|\Omega| = 2^n$

$\boxed{X = \#\text{ of heads}}$

$\Omega_X = \{0, 1, 2, \ldots, n\}$

$$p_X(k) = P(X = k) = \underbrace{\binom{n}{k}}_{\#\text{ seqs w/ } k \text{ H's}} \underbrace{p^k (1-p)^{n-k}}_{\text{Prob of a part seq w/ } k \text{ heads.}}$$

$0 \le k \le n$

T H H T T H T T

$(1-p)$ p p $(1-p$

$p^3 (1-p)^{n-3}$

$$p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,2,\dots,n \\ 0 & \text{otherwise.} \end{cases}$$

Example – Number of Heads

We flip n coins, independently, each heads with probability p

$\Omega = \{\mathrm{HH}\cdots\mathrm{HH}, \mathrm{HH}\cdots\mathrm{HT}, \mathrm{HH}\cdots\mathrm{TH}, \dots, \mathrm{TT}\cdots\mathrm{TT}\}$

$X =$ # of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

# of sequences with k heads

Prob of sequence w/ k heads

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Expectation (Idea)

Example. Toss a coin 20 times independently with probability ¼ of coming up heads on each toss.

X = number of heads

How many heads do you ***expect*** to see?

5

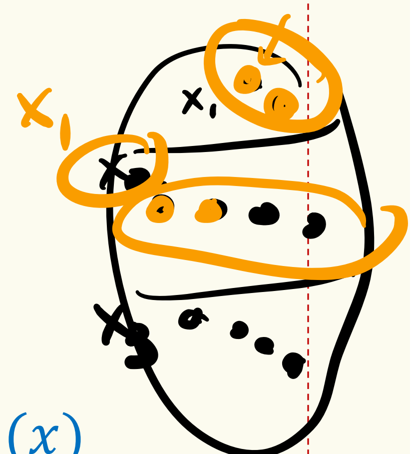
What if you toss it independently n times and it comes up heads with probability p each time?

np

Review Expected Value of a Random Variable

Definition. Given a discrete RV $X : \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$



or equivalently

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x) \quad = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Expectation

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\mathbb{E}[X] = \sum_{x \in \Omega_X} x \cdot P(X = x)$$

Example. Two fair coin flips

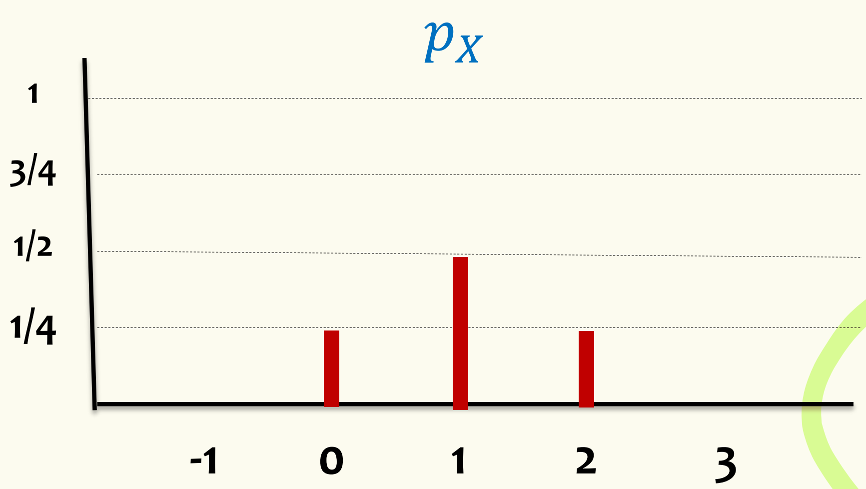
$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

$X =$ number of heads

What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = X(TT)\,P(TT) + X(HT)\,P(HT) + X(TH)\,P(TH) + X(HH)\,P(HH)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} = 1$$



$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

3 $\frac{1}{2}$

Example: Returning Homeworks

$$\boxed{\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)}$$

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x)$$

Ω_X

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW
- What is $\mathbb{E}[X]$?

$\Pr(\omega)$	ω	$X(\omega)$
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1/6	2, 1, 3	1
1/6	1, 2, 3	3

$$E(X) = 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 3 \frac{1}{6} = 1$$

$$E(X) = 0 \cdot P(X=0) + 1 \cdot P(X=1) + 3 P(X=3)$$

Example: Returning Homeworks

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

$$\boxed{\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x)}$$

Ω_X

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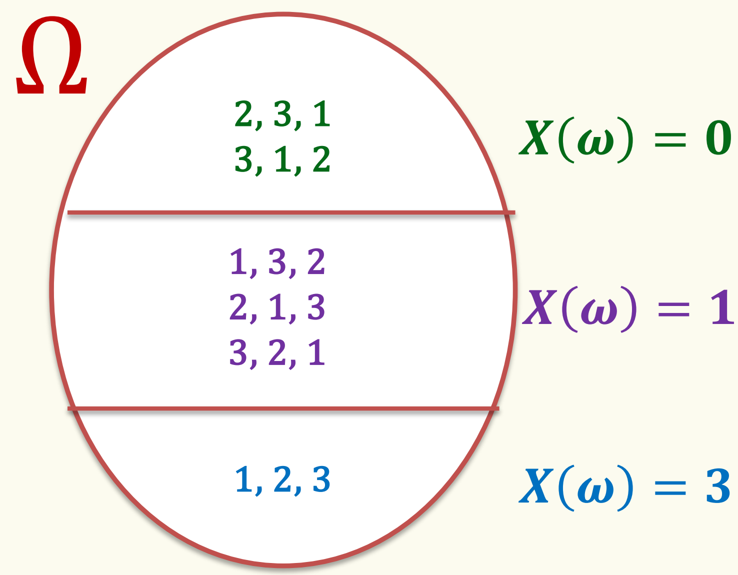

Example: Returning Homeworks

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$1/6$	$2, 3, 1$	0
$1/6$	$3, 1, 2$	0
$1/6$	$3, 2, 1$	1

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= 6 \cdot \frac{1}{6} = 1$$

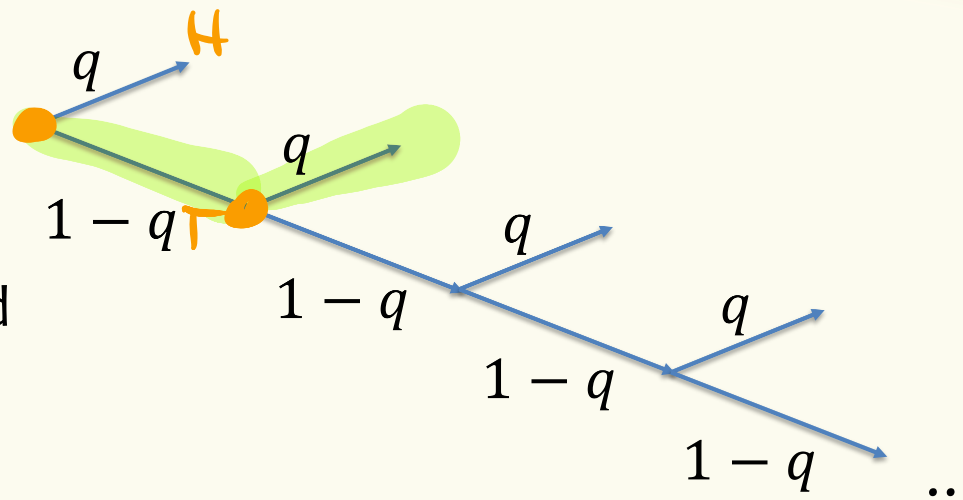
Example – Flipping a biased coin until you see heads

- Biased coin, each flip indep:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- Z = # of coin flips until first head



$P(Z = 0) = 0$

$P(Z = 1) = q$

$P(Z = 2) = (1-q)q$

$\boxed{P(Z = i) = (1-q)^{i-1}q}$

$\Omega_X = \{1, 2, 3, \ldots\}$

$\mathbb{E}[Z] = \sum_{x \in \Omega_Z} x P(Z = x) = \sum_{i=1}^{\infty} i \underbrace{P(Z = i)}$

$= \sum_{i=1}^{\infty} i (1-q)^{i-1} q$

$i=1$

Example – Flipping a biased coin until you see heads

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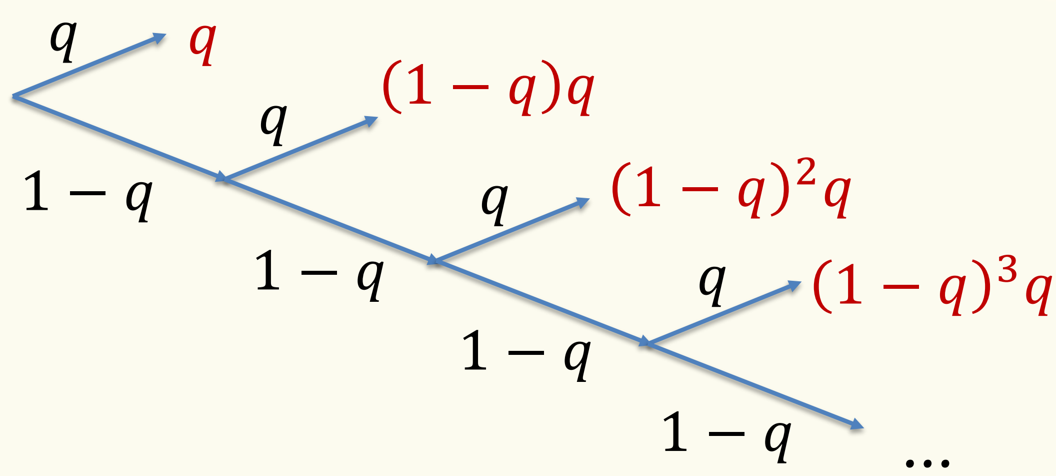


$$P(Z = i) = q\,(1-q)^{i-1}$$

$$\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot P(Z = i) = \sum_{i=1}^{\infty} i \cdot q(1-q)^{i-1}$$

Converges, so $\mathbb{E}[Z]$ is finite

Can calculate this directly but…

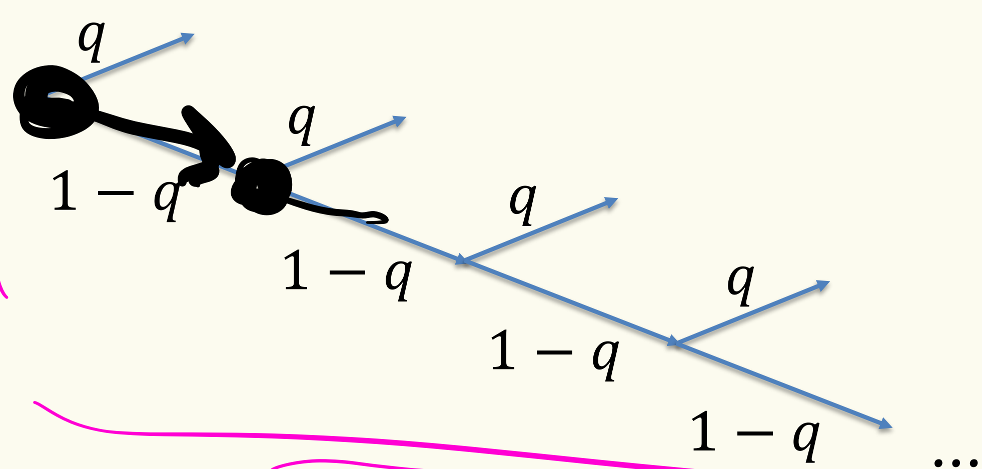
$q = \frac{1}{20}$

Example – Flipping a biased coin until you see heads

- Biased coin, each flip indep:

  $P(H) = q > 0$

  $P(T) = 1 - q$
- Z = # of coin flips until first head



Another view: If you get heads first try you get $Z = 1$;
If you get tails you have used one try and have the same experiment left

$$\mathbb{E}[Z] = 1 \cdot q + (1-q)(1 + E(Z))$$

$$\boxed{E(Z) = 1 + (1-q)E(Z)}$$

$$qE(Z) = 1$$

$\Rightarrow E(Z) = \frac{1}{q}$

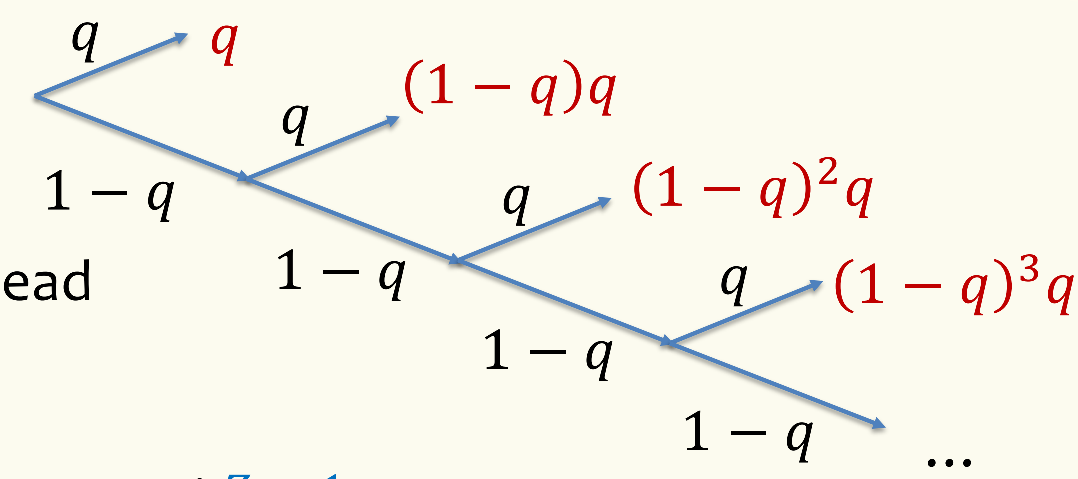
Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- Z = # of coin flips until first head



Another view: If you get heads first try you get $Z = 1$;
If you get tails you have used one try and have the same experiment left

$$\mathbb{E}[Z] = q \cdot 1 + (1-q)(1 + \mathbb{E}(Z))$$

Solving gives $q \cdot \mathbb{E}[Z] = q + (1-q) = 1$ Implies $\mathbb{E}[Z] = 1/q$

Example – Coin Tosses

We flip n coins, each toss independent, probability p of coming up heads.

Z is the number of heads, what is $\mathbb{E}(Z)$?

$$E(Z) = \sum_{k=0}^{n} k\, P(\underline{Z = k})$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

Example – Coin Tosses

We flip n coins, each toss independent; heads with probability p,

Z is the number of heads, what is $\mathbb{E}[Z]$?

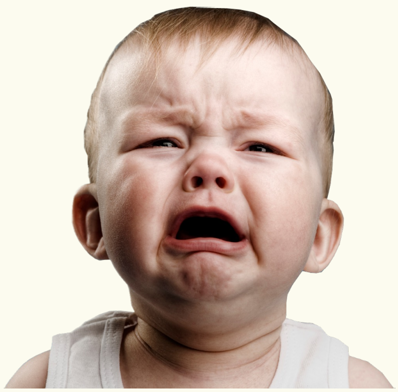
$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) \quad = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} \quad = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$



Can we solve it more elegantly, please?

Agenda

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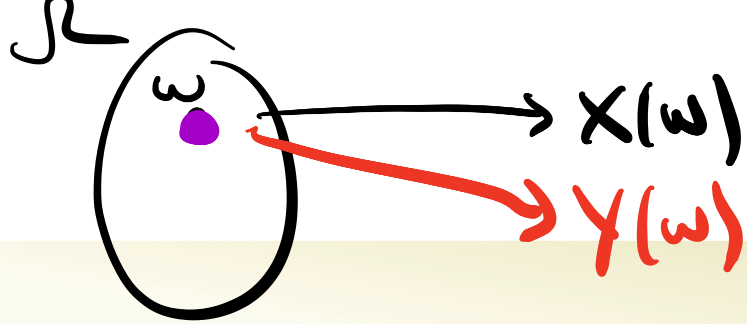
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation
- Linearity of Expectation

$Z = X + Y$



Linearity of Expectation

$Z(\omega) = X(\omega) + Y(\omega)$

Theorem. For any two random variables X and Y

(no conditions whatsoever on the random variables)

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Because: $\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[(X_1 + \cdots + X_{n-1}) + X_n]$

$$= \mathbb{E}[X_1 + \cdots + X_{n-1}] + \mathbb{E}[X_n] = \cdots$$

$Z = X^2 + 4Y$

$Z(\omega) = (X(\omega))^2 + 4Y(\omega)$

Linearity of Expectation – Proof

Theorem. For any two random variables X and Y

(X, Y do not need to be independent)

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$Z = X + Y$

$Z(\omega) = X(\omega) + Y(\omega)$

$$\mathbb{E}[X+Y] = \sum_\omega P(\omega)(X(\omega) + Y(\omega))$$

$$= \sum_\omega P(\omega)X(\omega) + \sum_\omega P(\omega)Y(\omega)$$

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

$E(\tilde{X}) = \sum_{\omega \in \Omega} \tilde{X}(\omega) P(\omega)$

Using LOE to compute complicated expectations

want $E(X)$

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

Often, X_i are **indicator** (0/1) random variables.

Kandinsky