CSE 312

Foundations of Computing II

Lecture 4: Intro to discrete probability

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for polls and anonymous questions

Probability

- We want to model uncertainty.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.
- We will not argue why a certain physical process realizes the probabilistic model we study
  - Why is the outcome of the coin flip really "random"?
- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Sample Space

Omega

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

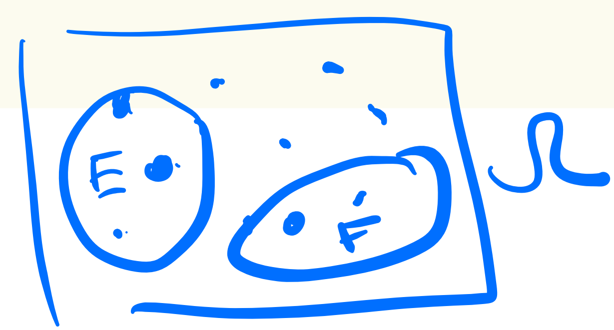
Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$



Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

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Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

$A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. D1 + D2 = 6

$B = \{(2,4), (3,3), (4,2)\}$

C. D1 = 2 * D2

$C = \{(2,1), (4,2)\}$

Die 2 (D2)

Die 1 (D1)

	1	**2**	**3**	**4**
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

Are A and B mutually exclusive?
How about B and C?

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}\colon \Omega \to [0,1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

- Also use notation: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing

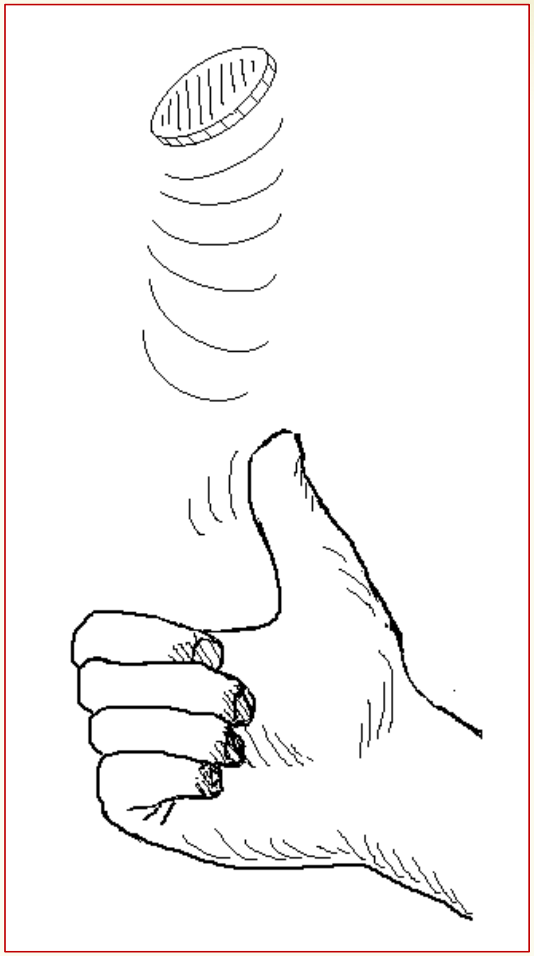
Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(\mathrm{H}) = \mathbb{P}(\mathrm{T}) = \frac{1}{2} = 0.5$$



Example – Coin Tossing

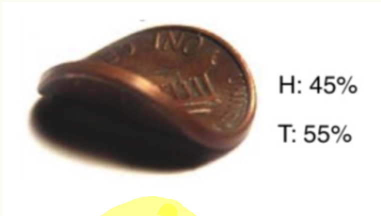
Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(\mathrm{H}) = 0.45, \qquad \mathbb{P}(\mathrm{T}) = 0.55$$



Probability space

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space**.
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Probability space

Either finite or infinite countable (e.g., integers)

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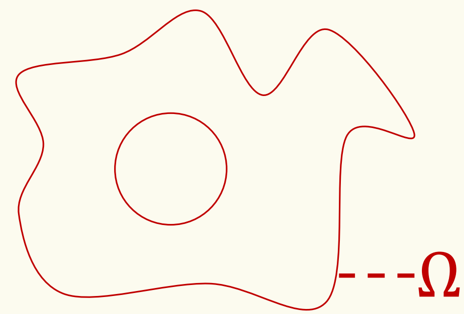
Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

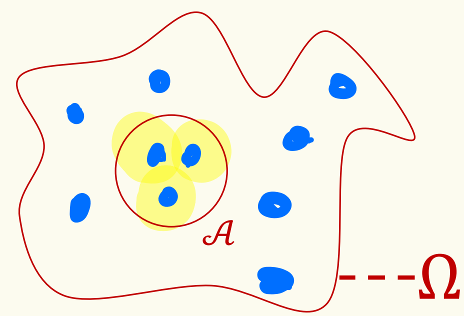
Examples:

- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$

Events

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



Convenient abuse of notation: $\mathbb{P}$ is extended to be defined over **sets**. $\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$

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Example: 4-sided Dice, Event Probability

$P(\omega) = \frac{1}{|\Omega|} = \frac{1}{16}$

Think back to 4-sided die. Suppose each outcome is equally likely.
What is the probability of event B? $\Pr(B) = ???$

B. D1 + D2 = 6

$B = \{(2,4), (3,3)(4,2)\}$

$\Pr(B) = \sum_{\omega \in B} \Pr(\omega)$

$= P(2,4) + P(3,3) + P(4,2)$

$= \frac{3}{16}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
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Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.

Example – Coin Tossing

$\Omega = \{\text{sequences of 100 coin flips}\} = \{H,T\}^{100}$

$|\Omega| = 2^{100}$ $\quad P(\omega) = \frac{1}{2^{100}}$

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

E: {sequences that contain exactly 50 Hs}

~~https://pollev.com/annakarlin~~185

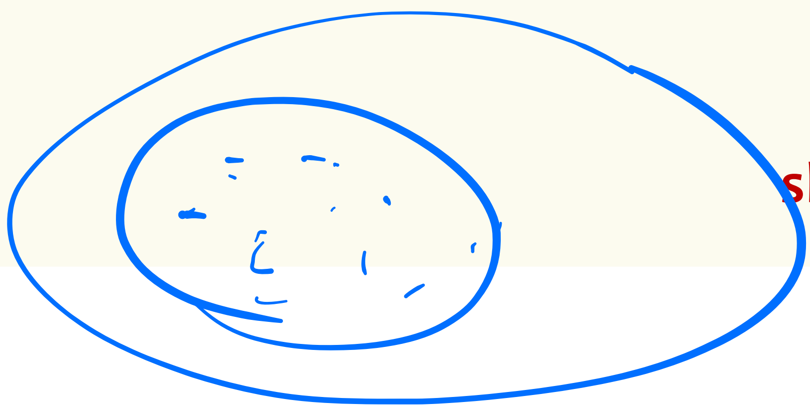
$\Pr(E) = \frac{|E|}{|\Omega|}$

(A) $\frac{1}{2}$

(B) $\frac{1}{2^{50}}$

(C) $\frac{\binom{100}{50}}{2^{100}}$

(D) Not sure



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Brain Break

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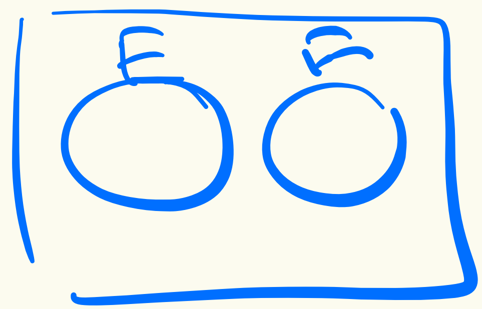
Axioms of Probability

P:

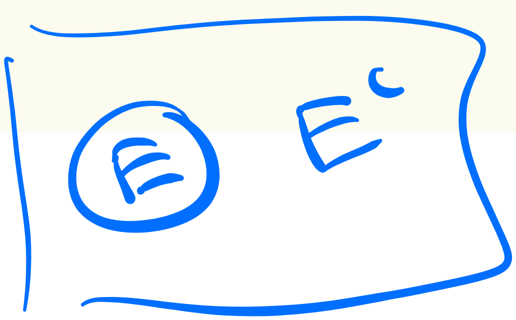
Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to **any** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If *E* and *F* are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$



$E^c = \bar{E}$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$



$\Omega = E \cup E^c$

$1 = P(\Omega) = P(E \cup E^c) = P(E) + P(E^c)$

Review Probability space

Either finite or infinite countable (e.g., integers)

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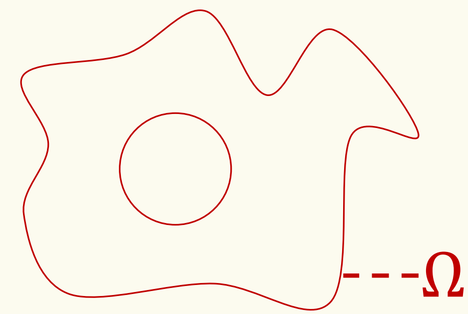
Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

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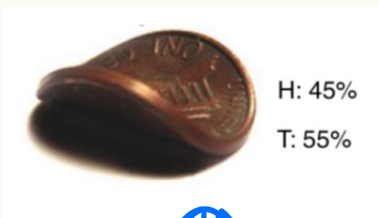
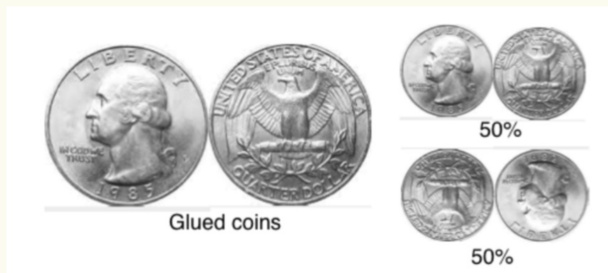
Non-equally Likely Outcomes

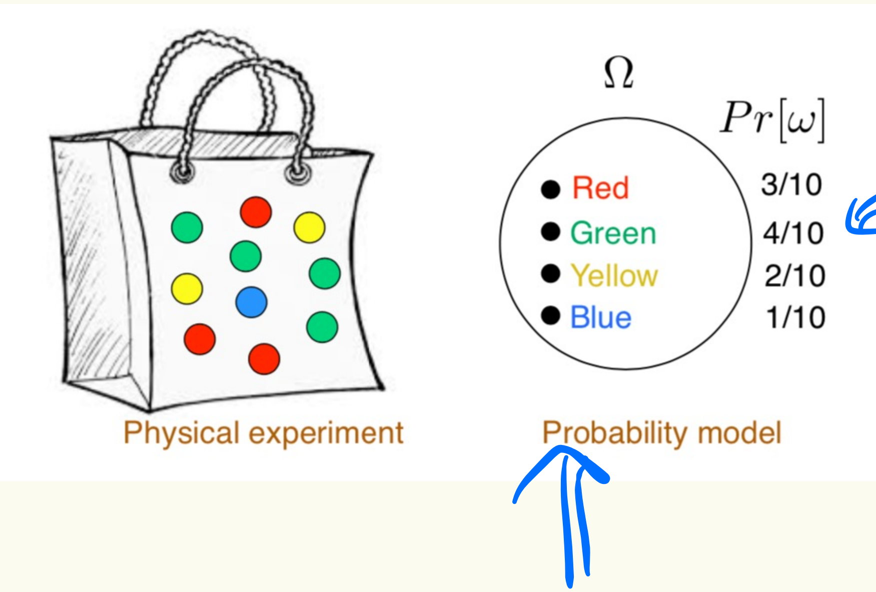
Probability spaces can have **non-equally likely outcomes.**

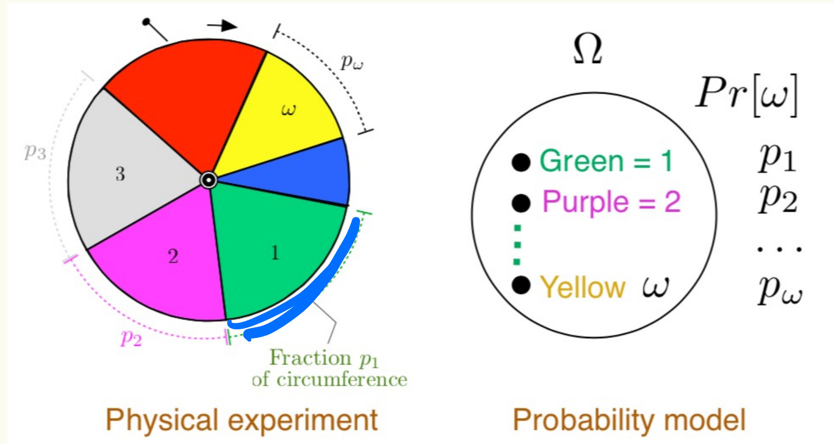




$$\Omega = \{\underset{0}{HH}, \underset{\frac{1}{2}}{HT}, \underset{\frac{1}{2}}{TH}, \underset{0}{TT}\}$$

More Examples of Non-equally Likely Outcomes





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Example: Dice Rolls

all outcomes equally likely.

Suppose I had a two, fair, 6-sided dice that we roll, one green, one red. What is the probability that we see *at least one 3 in the two rolls*.

$|\Omega| = 36$ $E = \{\text{at least one } 3\}$

$Pr(E) = 1 - P(E^c)$

$P(\text{see no 3's}) = \frac{25}{36}$

$= \frac{11}{36}$

Example: Birthday "Paradox"

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assume that there are exactly 365 possible birthdays, and each possibility is equally likely.

A B C ...

$\Omega = \{$ Jan 1 Jan 1 Jan 1, Jan 1 Jan 2 Jan 3 $\}$

$|\Omega| = 365^n$

$Pr(w) = \frac{1}{365^n}$

$w = \{$ Jan 3, Mar 10, Feb 12, Apr 3 ... $\}$

$E = \{$ at least 2 people share a bday $\}$

$Pr(E) = 1 - Pr($ no two people share a bday $)$ (E^c)

n=4

A B C D

365 364 363 362

365−0 365−1 365−2 365−3

only correct $n \le 365$

Example: Birthday "Paradox" cont.

$$Pr(E^c) = \frac{|E^c|}{|\Omega|} = \frac{365 \cdot 364 \cdot 363 \cdots (365-n+1)}{365^n}$$

365 − (n−1)

$$= \frac{365!}{(365-n)!\, 365^n}$$

n = 23 P(E) > 0.5

n = 60 P(E) > 0.99.

Pr(at least one of n people has bday Jan 1)

1 − Pr(none have Jan 1)

$\frac{364^n}{365^n}$

n=23 Prob 0.06

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.

students 1 2 3

Outcomes
1, 2, 3
1, 3, 2
2, 1, 3
2, 3, 1
3, 1, 2
3, 2, 1

$\Omega =$ perm's of n elts

$|\Omega| = n!$

$P(w) = \frac{1}{n!}$

Pr(person 17 gets their own hw back)

$= \frac{|\text{perms with a 17 in pos 17}|}{n!}$

$$= \frac{(n-1)!}{n!} = \frac{1}{n}$$

$n-1$

↓

__ __ __ __ $\frac{17}{17}$ __ __ __ __