CSE 312

Foundations of Computing II

Lecture 3: More counting!

PAUL G. ALLEN SCHOOL OF COMPUTER SCIENCE & ENGINEERING

Anna R. Karlin

Recap (1)

Product Rule: In a sequential process, there are

- n_1 choices for the first step,
- n_2 choices for the second step (given the first choice), … , and
- n_m choices for the m^{th} step (given the previous choices),

then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$

Application. # of k-element sequences of distinct symbols (a.k.a. k-permutations) from n-element set is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Recap (2)

Combination: If order does not matter, then count the number of ordered objects, and then divide by the number of orderings

Applications. The number of subsets of size k of a set of size n is

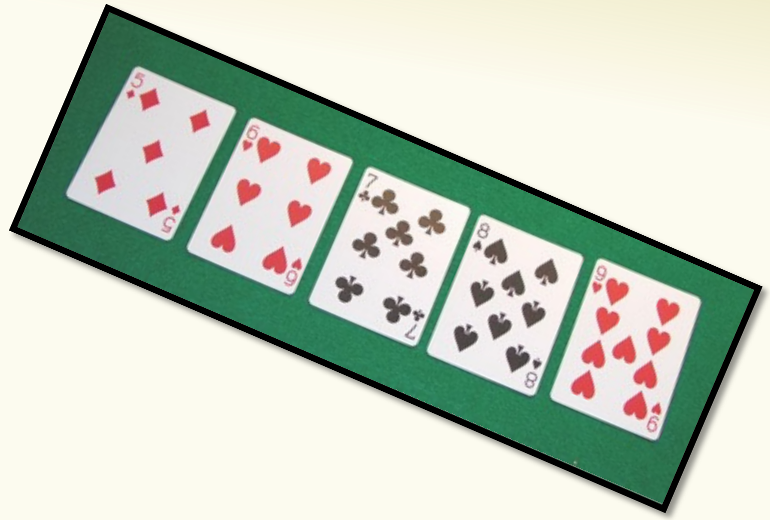
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Agenda

- More Examples + Sleuth’s Criterion
- Stars and Bars
- Pigeonhole Principle
- Combinatorial Proofs

Quick Review of Cards



How many possible 5 card hands?

$\binom{52}{5}$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

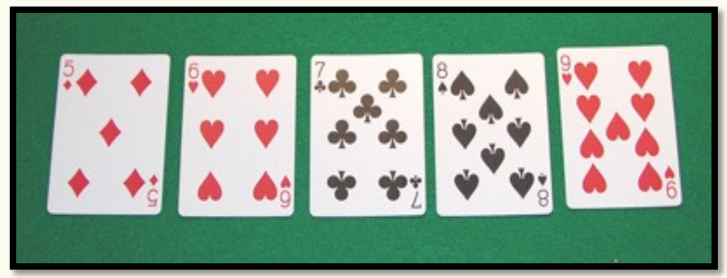
Counting Cards I

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A **straight** is five consecutive rank cards of any suit. How many possible straights?

- choose lowest rank
- choose suits fo

$10 \cdot 4^5$



A 2 3 4 5 6 7 8 9 10 J Q K A

Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is a five card hand all of the same suit.

How many possible flushes?

- suit — 4
- any 5 of that suit — $\binom{13}{5}$

$$4 \cdot \binom{13}{5}$$

Counting Cards III

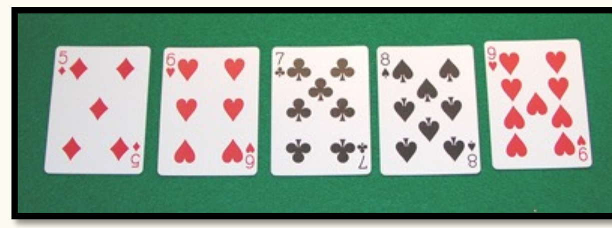
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

A **flush** is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

How many flushes are NOT straights?



\# flushes − # straight flushes.

$$4 \cdot \binom{13}{5} - 10 \cdot 4$$

Counting Cards III

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

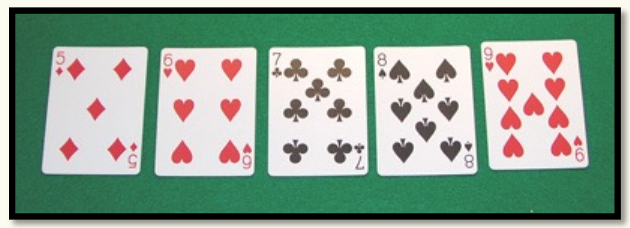
- A flush is five card hand all of the same suit. How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

- How many flushes are NOT straights?

= #flush - #flush and straight

$$\left(4 \cdot \binom{13}{5} = 5148\right) - 10 \cdot 4$$



Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting

Many sequences ➔ over counting

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence → under counting   Many sequences → over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. $\binom{4}{3} \cdot \binom{49}{2}$

Poll:

A. Correct
B. Overcount
C. Undercount

https://pollev.com/ annakarlin185

Sleuth's Criterion (Rudich)

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No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

A? AC AD 2H KS

A? AC AD AS KS

Counted 4 x.

excess: 3 · (# hands w/ 4 Aces)

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

hands w/ 3 Aces $\binom{4}{3}\binom{48}{2}$

When in doubt, break up into disjoint sets you know how to count, and then use the sum rule.

hands w/ 4 Aces 48

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence ➔ under counting   Many sequences ➔ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

= # 5 card hand containing exactly 3 Aces — $\binom{4}{3} \cdot \binom{48}{2}$

+ # 5 card hand containing exactly 4 Aces — $\binom{48}{1}$

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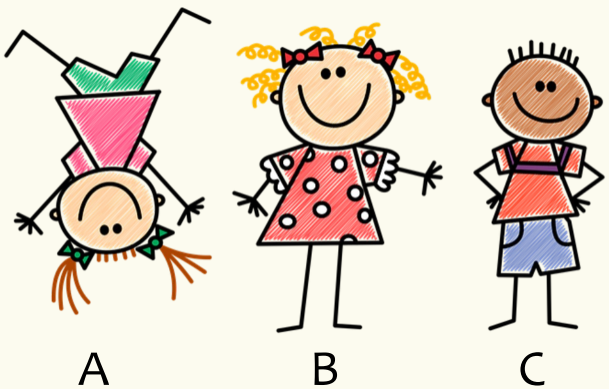
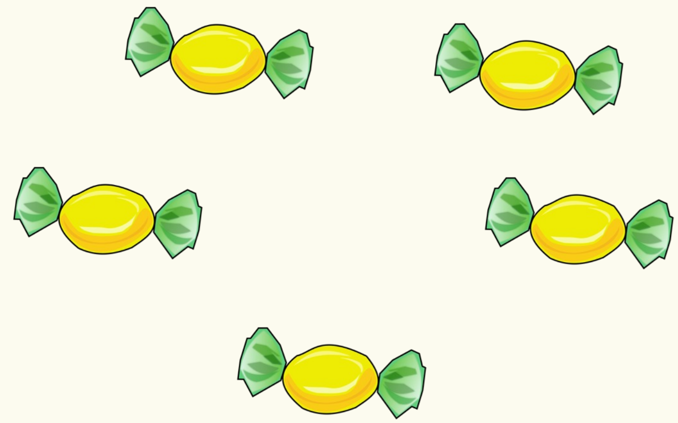
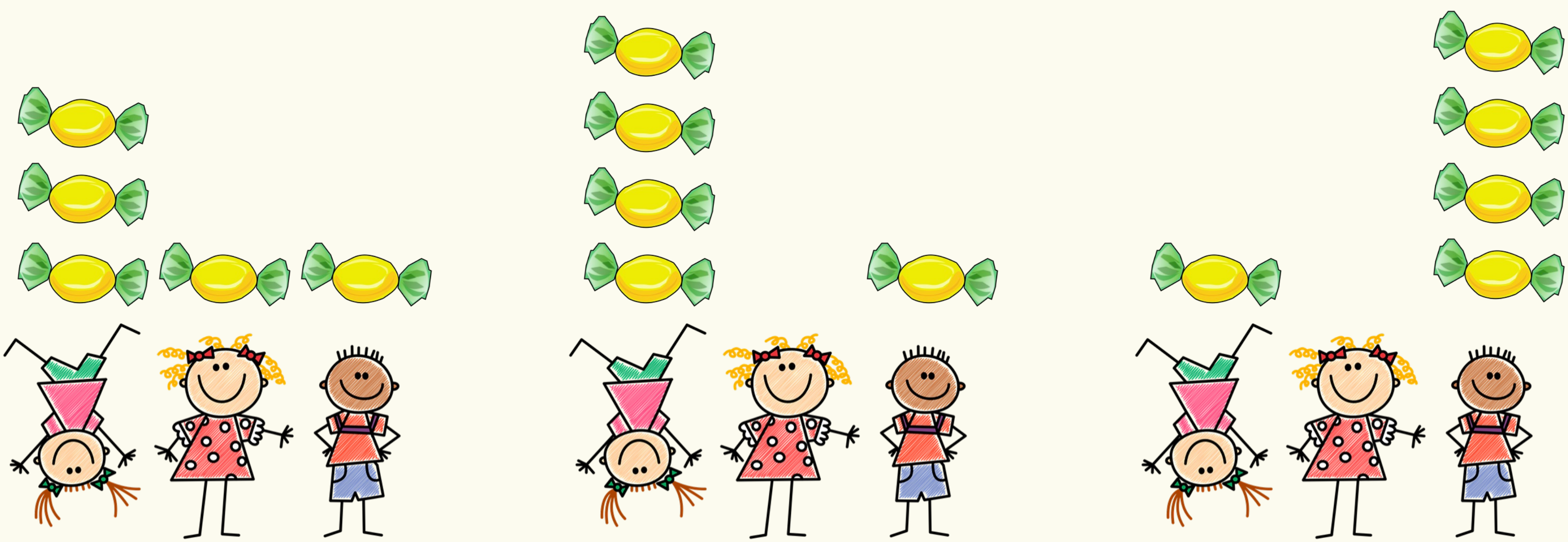
Example: Kids and Candies





How many ways can we give five **indistinguishable** candies to these three kids?

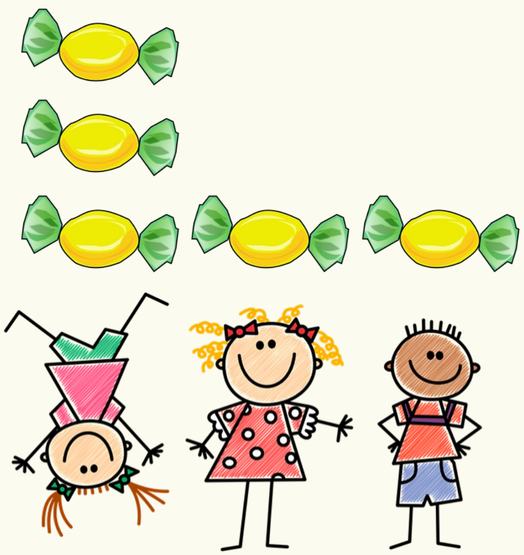
Kids + Candies



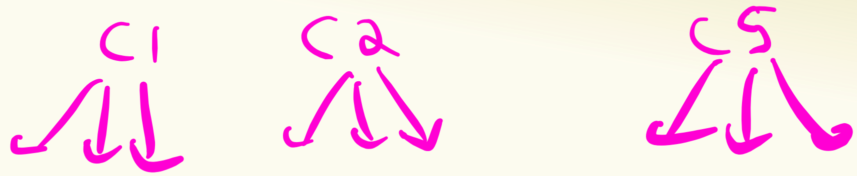
Kids + Candies

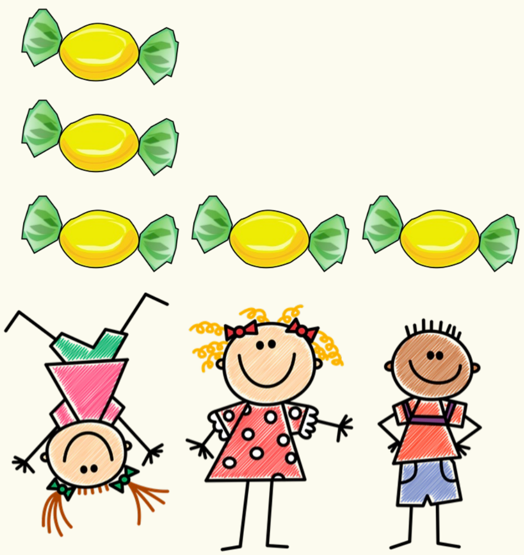
- First try: first choose how many candies kid 1 gets, then how many kid 2 gets, etc.

6



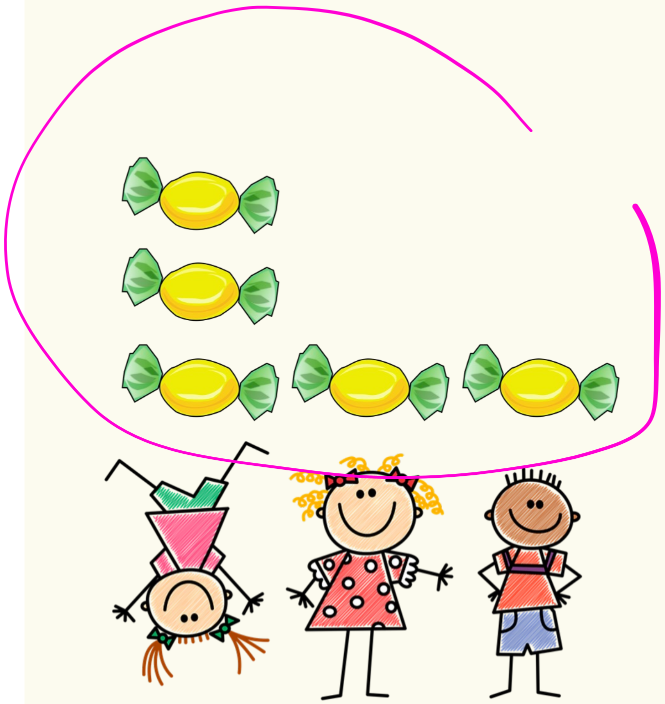
Kids + Candies





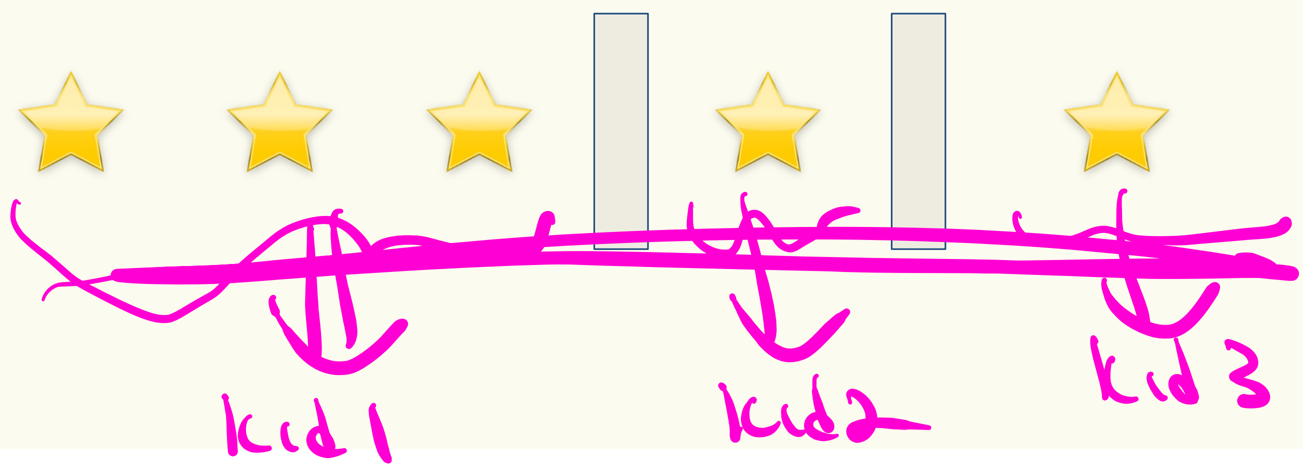
- Second try: lay down the 5 candies in a row. First choose which kid gets candy 1, then which kid gets candy 2, and so on.
- How many times is the outcome where kid 1 gets all 5 candies counted?
- How many times is the outcome where kid 1 gets 4 and kid 2 gets 1 counted?

Kids + Candies

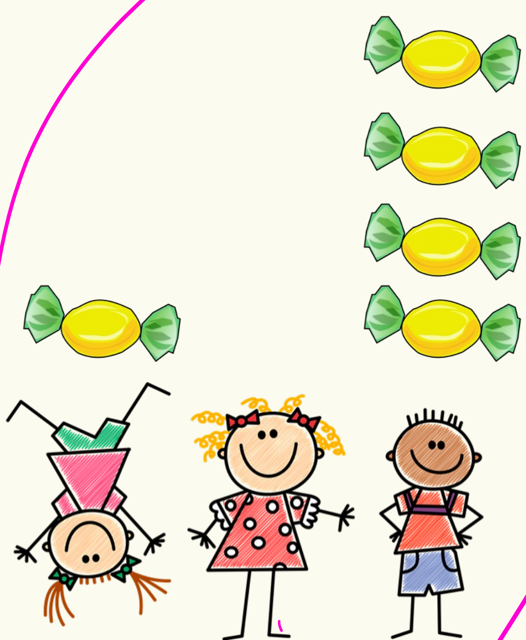


Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.



Kids + Candies



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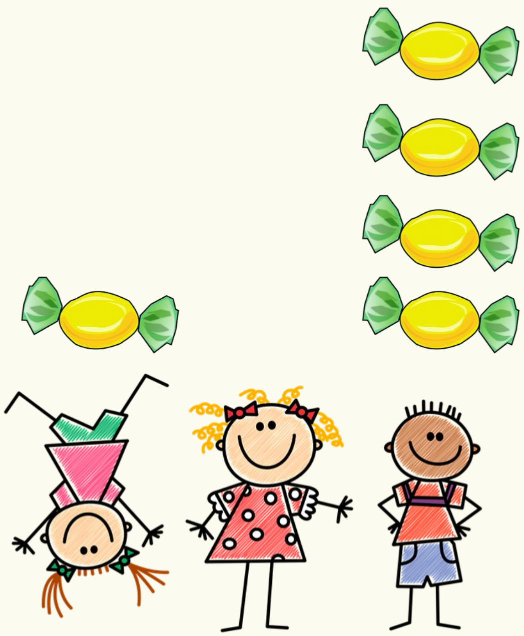
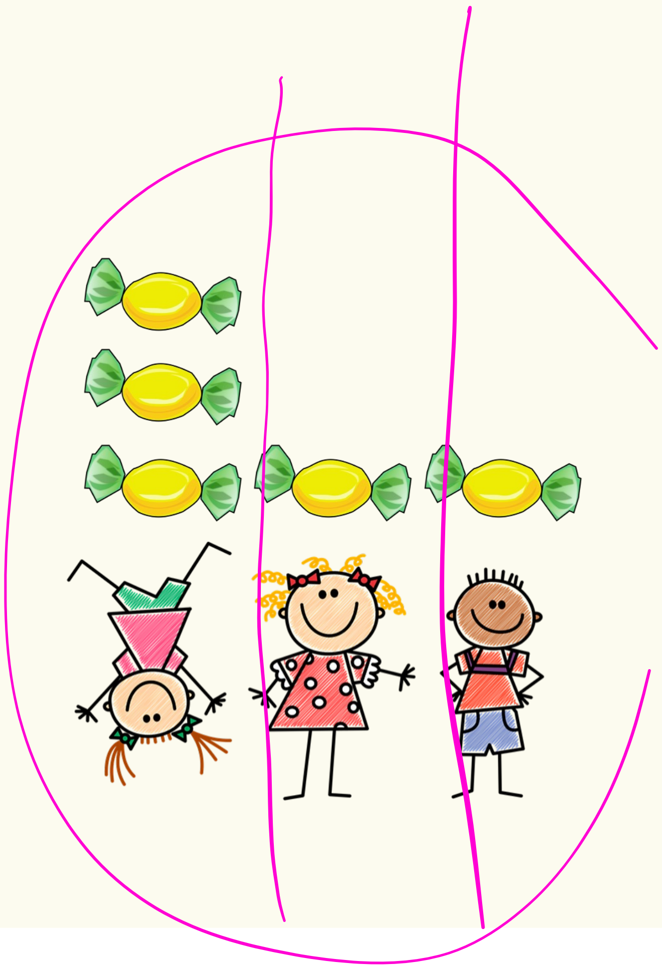
Kids + Candies



For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

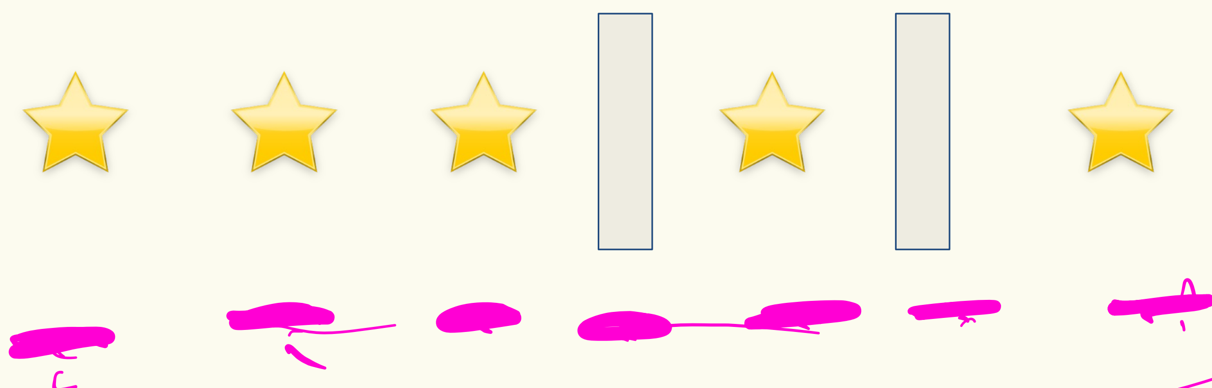
Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Kids + Candies

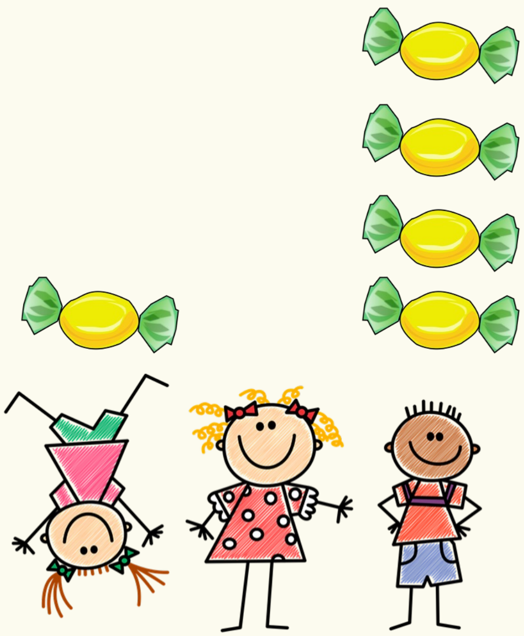


How many ways to construct a sequence with 5 stars and 2 bars?

$$\binom{7}{2} = \binom{7}{5}$$



Kids + Candies



Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

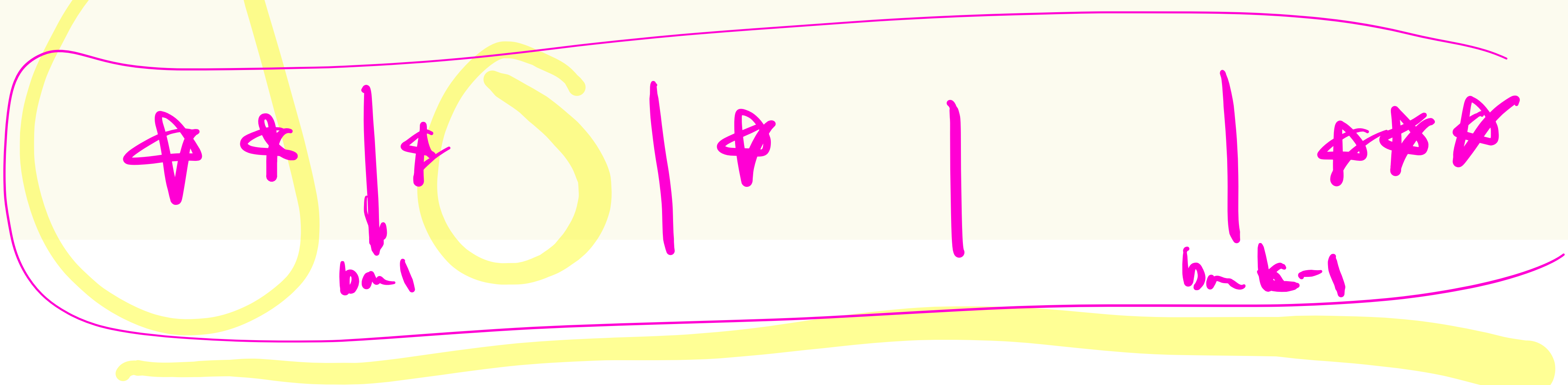
This is

$$\binom{7}{2} = \binom{7}{5}$$

Stars and Bars / Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$



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- Pigeonhole Principle
- Combinatorial Proofs

Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes

Pigeonhole Principle – More generally

If there are n pigeons in $n - 1$ holes, then one hole must contain at least 2 pigeons!

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k\frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

10 pigeons → 3
$\left\lceil \frac{10}{3} \right\rceil = \left\lceil 3\frac{1}{3} \right\rceil = 4$

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **366** holes = possible birthdays (because of leap year)
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps:

1. Identify pigeons
2. Identify pigeonholes
3. Specify a rule for assigning pigeons to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***three*** *elements whose (pairwise) difference is a multiple of 37.*

7, 10523, −54, 10^{150}, ...

$i - j \equiv 0 \mod 37$

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

integers in S

0 mod 37, 1 mod 37, ... 36 mod 37

$i \in S \longrightarrow i \mod 37$

100 pg → 37 PHs

∃ hole w/ at least $\left\lceil \frac{100}{37} \right\rceil = 3$ #'s

i, j, k

$i \mod 37 = j \mod 37 = k \mod 37$

$i - j \equiv 0 \mod 37$

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Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ Pascal's Identity

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ Follows from Binomial theorem

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ \textit{years later} \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

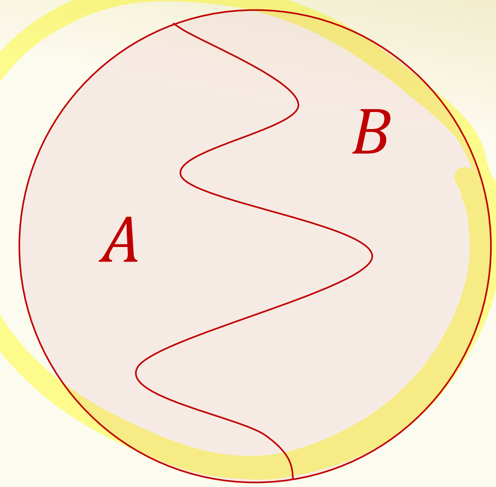
Hard work and not intuitive

Let's see a combinatorial argument

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$



$S = A \cup B$, disjoint

S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

A: the set of size k subsets of $[n]$ including n

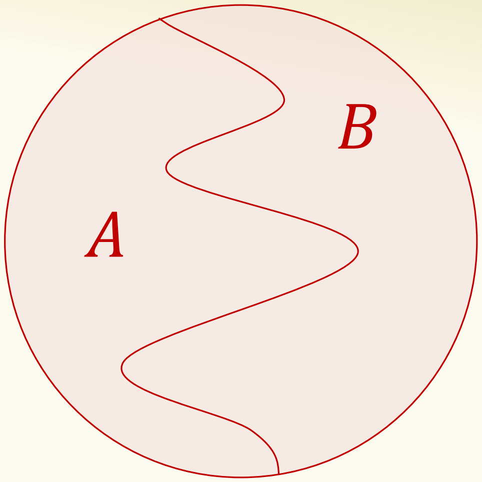
B: the set of size k subsets of $[n]$ NOT including n

Sum rule:
$|A \cup B| = |A| + |B|$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| = |A| + |B|$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, , k = 2, \quad S = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

A: the set of size k subsets of $[n]$ including n

$A = \{\{1,4\},\{2,4\},\{3,4\}\}. \qquad n = 4, k = 2$

B: the set of size k subsets of $[n]$ NOT including n

$B = \{\{1,2\},\{1,3\},\{2,3\}\}. \qquad n = 4, k = 2$

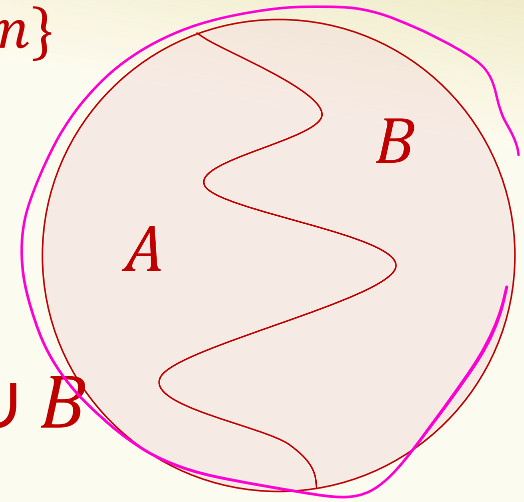
Example – Binomial Identity

$[n] = \{1, 2, \cdots, n\}$

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \qquad |A| \qquad |B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

How many?

A: the set of size k subsets of $[n]$ including n — $\binom{n-1}{k-1}$

B: the set of size k subsets of $[n]$ NOT including n — $\binom{n-1}{k}$

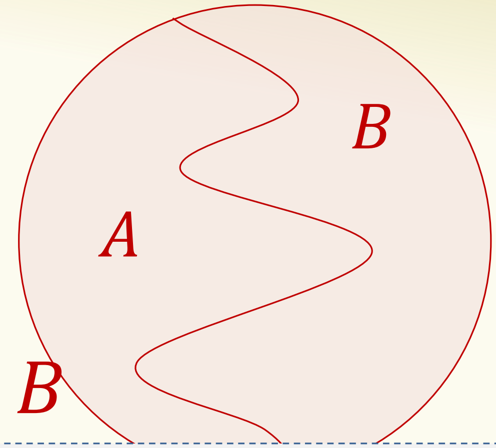
$\{1, 2, 3, \ldots, n-2, n-1, n\}$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \qquad |A| \qquad |B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$

A: the set of size k subsets of $[n]$ including n

n is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

B: the set of size k subsets of $[n]$ NOT including n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

Pascal's triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

$\binom{0}{0}$
$\binom{1}{0}$ $\binom{1}{1}$
$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$
$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$
$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$
$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$

$$(x+y)^4 = \sum x^i y^{4-i}$$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive