CSE 312

Foundations of Computing II

Lecture 15: Continuous RV

Midterm: Monday Feb 12, 1:30pm

Read my edstem post

Review Friday?

Bring ID

Locations:

Locations	Last Name
ECE 125	A-F
EXED 110	G-K
GUG 220	L-Z

Agenda

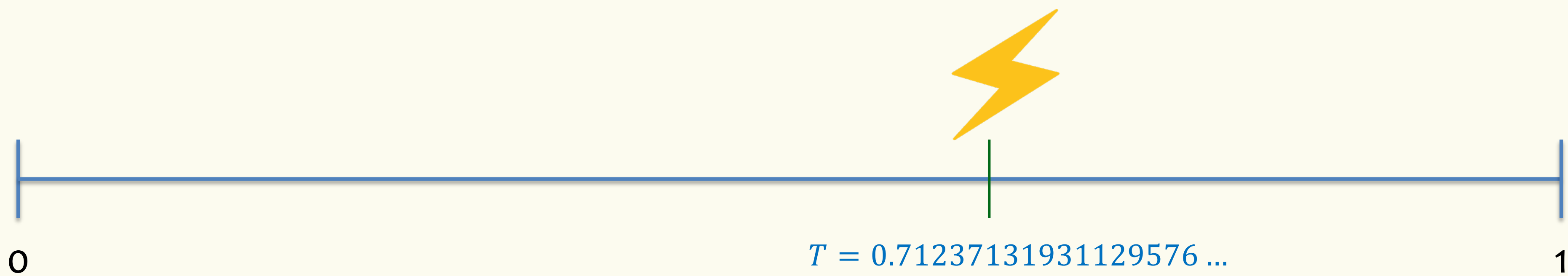
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance of continuous r.v.
- Introduction to continuous zoo

Often we want to model experiments where the outcome is not discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - – Time measured with infinitesimal precision.

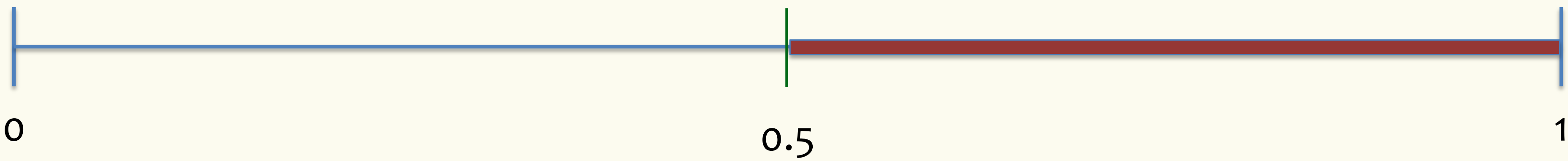


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T \geq 0.5) =$$



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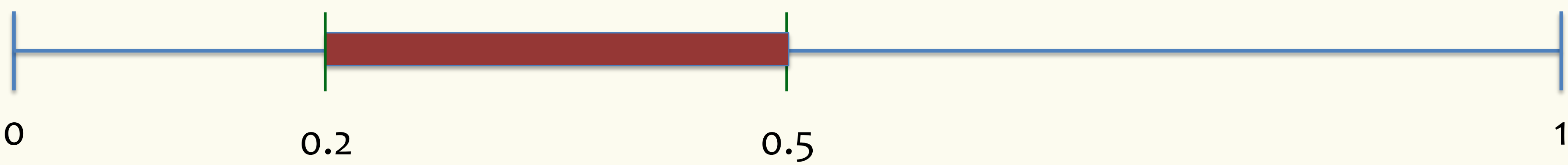
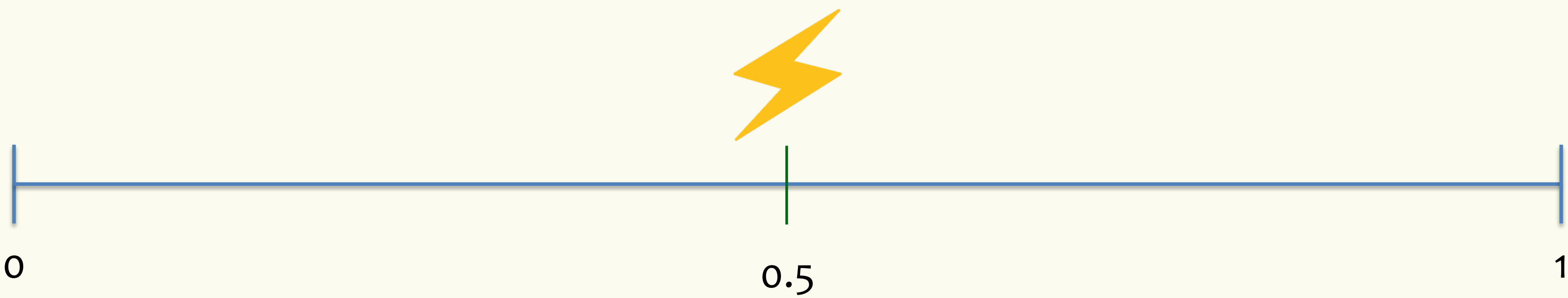


$P(0.2 \leq T \leq 0.5) =$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within [0,1] is equally likely



$P(T = 0.5) =$

Bottom line

- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $P(T \in [0,1]) = 1$
  - $P(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of T?

First try: A discrete approximation

Example – Lightning Strike

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- X = time of lightning strike
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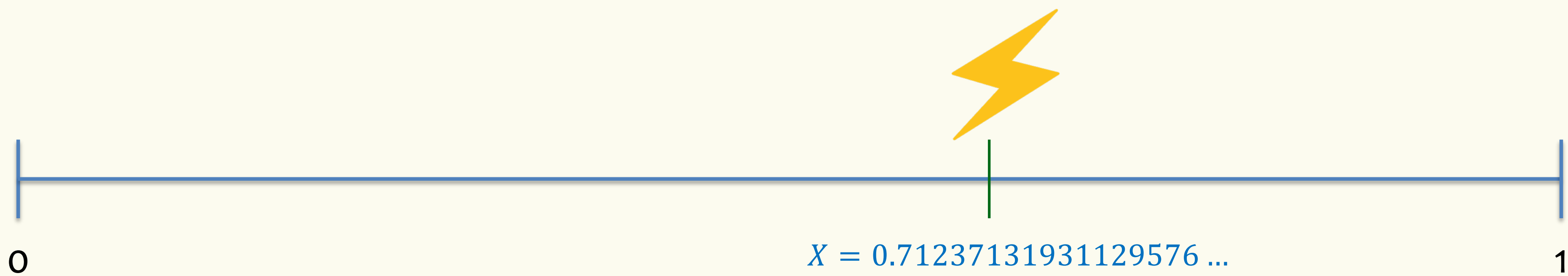
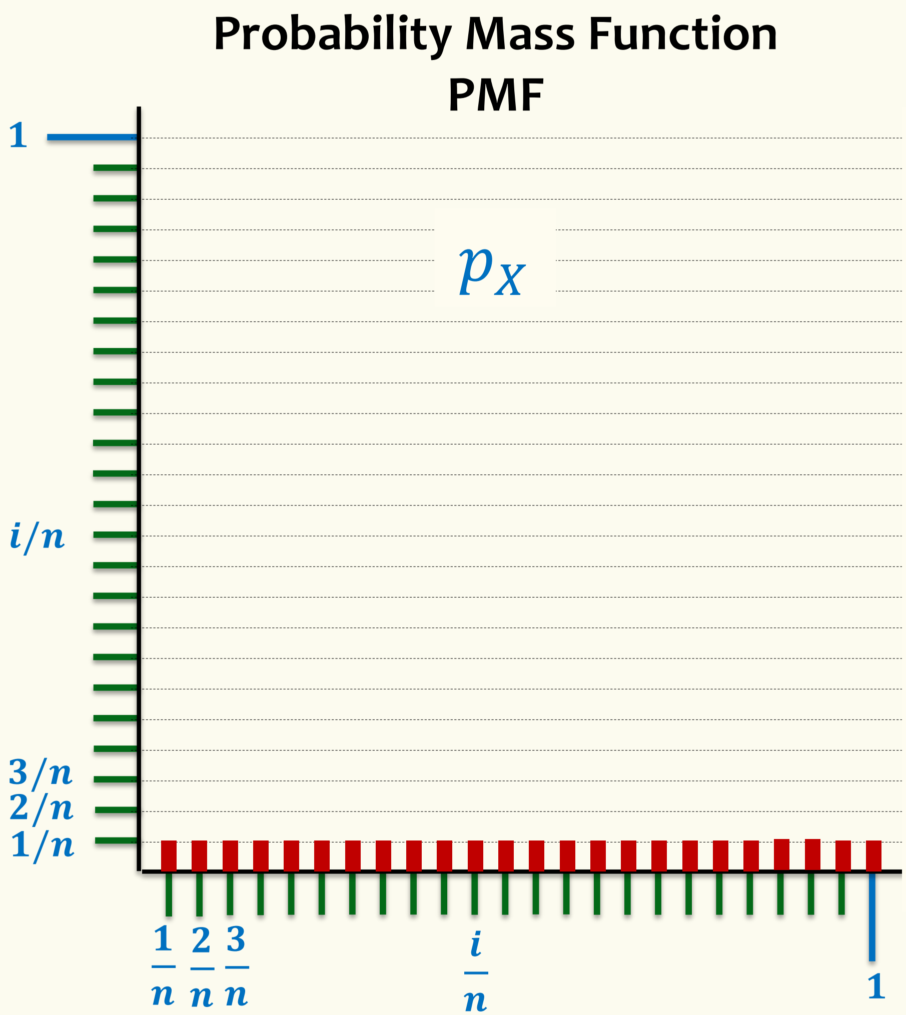


Discrete approximation?

A Discrete Approximation



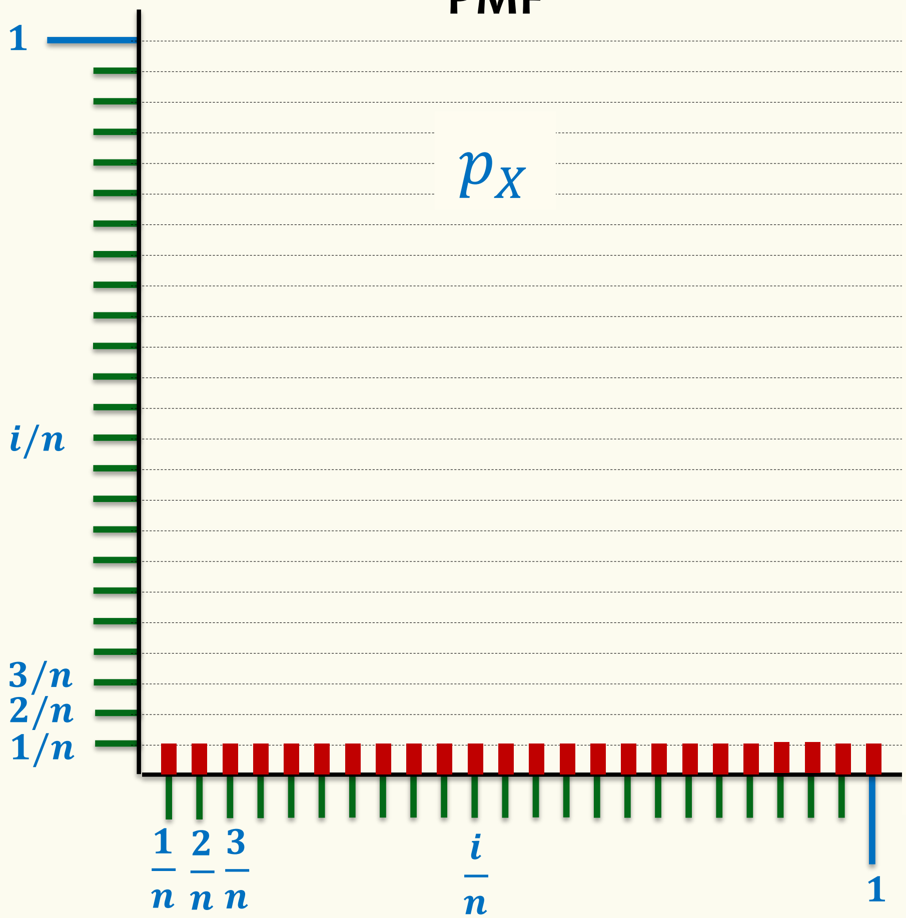
$$p_X\left(\frac{i}{n}\right) = \frac{1}{n} \qquad i = 1, \ldots, n$$

$$\lim_{n \to \infty}$$

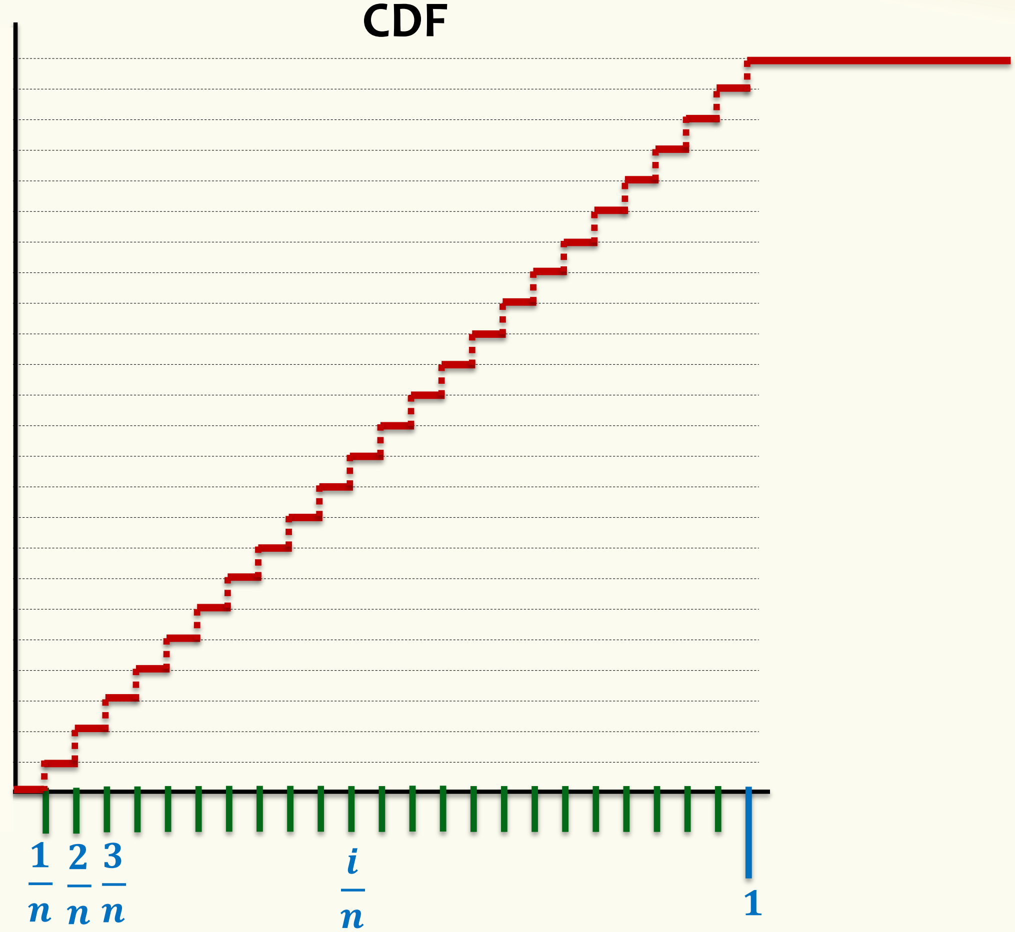
$F_X(x) = P(X \le x)$

A Discrete Approximation

Probability Mass Function
PMF



Cumulative Distribution Function
CDF



$$\lim_{n\to\infty} F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \end{cases}$$

Recall: Cumulative Distribution Function (CDF)

$p_X(x) = P(X = x)$

$\sum_{x \in \Omega_X} p_X(x) = 1$

$p_X(x) \geq 0$

$F_X(x) = P(X \leq x)$

$F_X \nearrow$ monotonically $0 \to 1$

Probability Mass Function

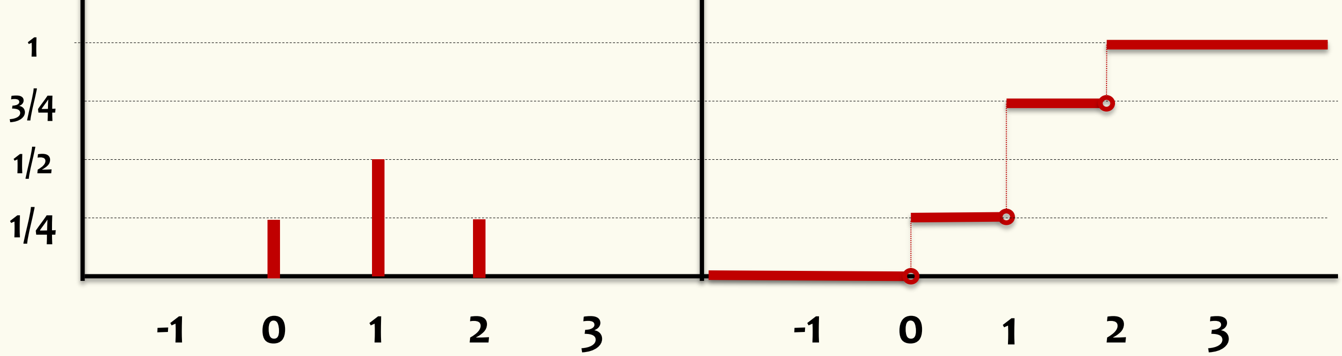
PMF

p_X

Cumulative Distribution Function

CDF

F_X



$\lim_{\varepsilon \to 0} \frac{F(x+\varepsilon) - F(x)}{\varepsilon} = \frac{d}{dx} F(x)$

$p_X(k) = \frac{F_X(k) - F_X(k-1)}{k - (k-1)}$

$F_X(x) = \sum_{z \leq x} p_X(z)$

$K - (K-1)$

$z \in \Omega_X$

let distance between values in $\Omega_X \to 0$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

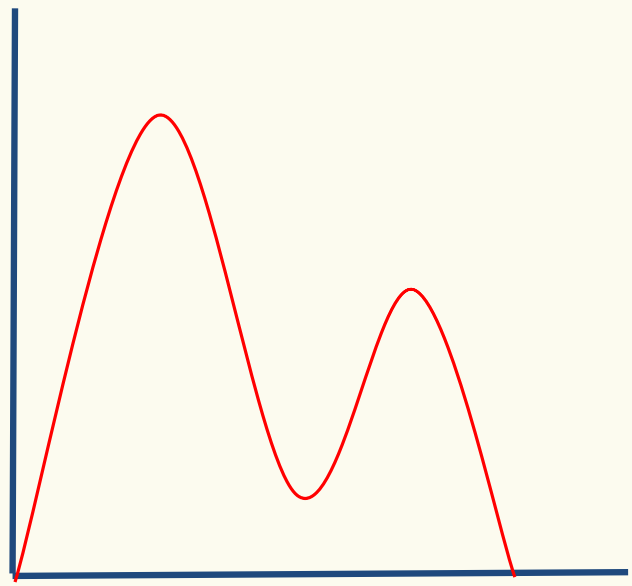
prob density fn pdf

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$= P(X \le x)$$

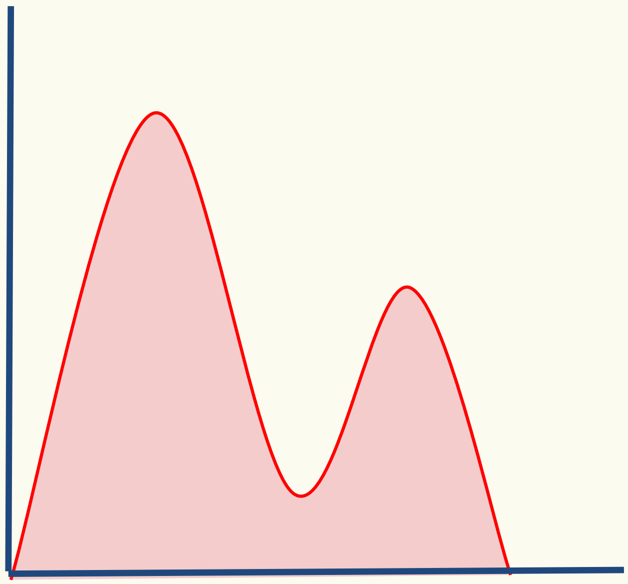
Capturing rate at which prob is accumulating at pt x

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

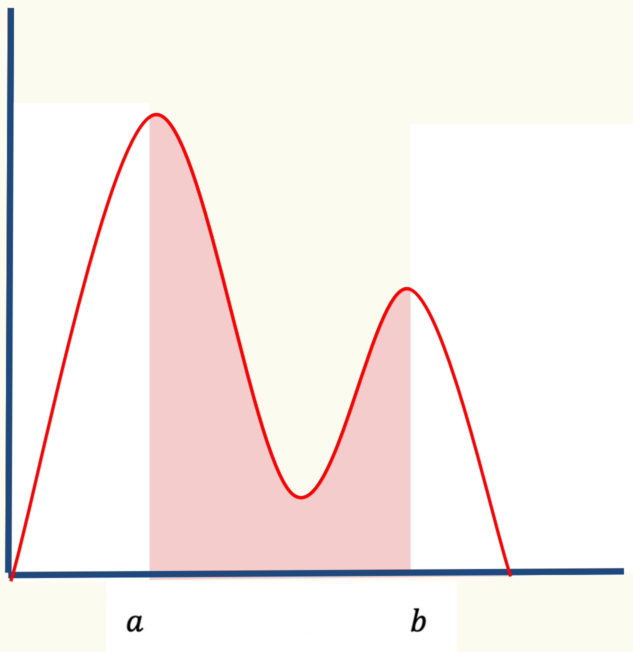
Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

Probability Density Function - Intuition



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$$F(b) - F(a) = P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

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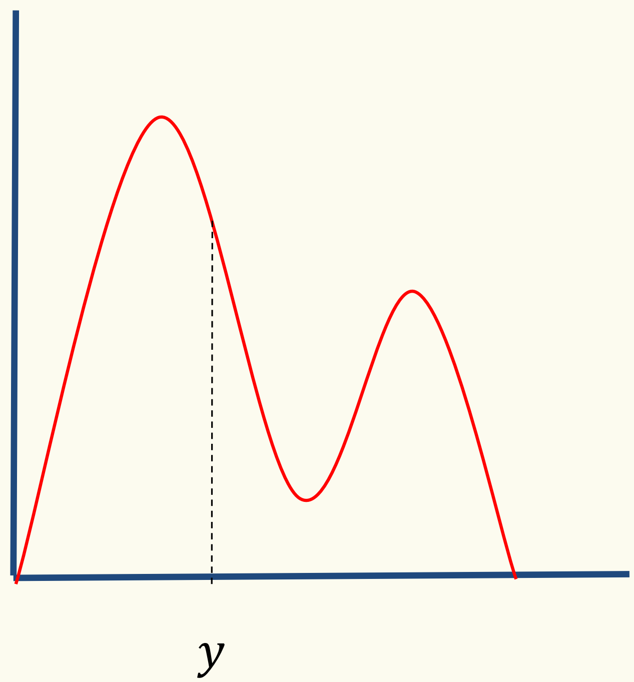

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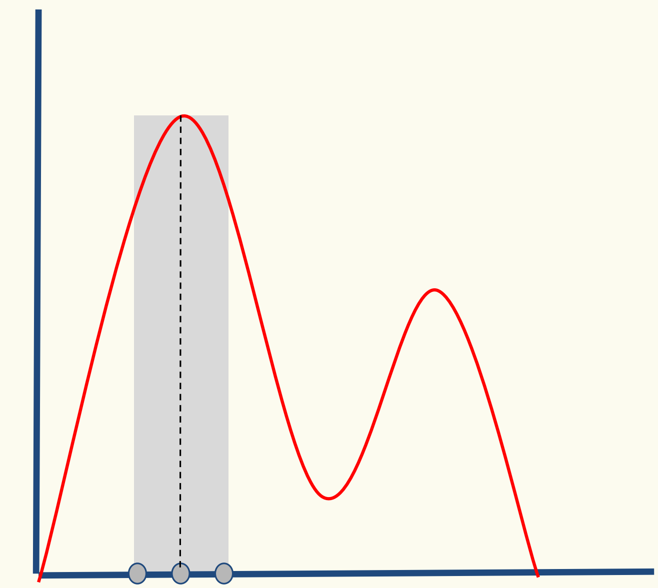
$$F(b) - F(a) = P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

Density ≠ Probability

$f_X(y) \neq 0 \qquad P(X = y) = 0$

Probability Density Function - Intuition



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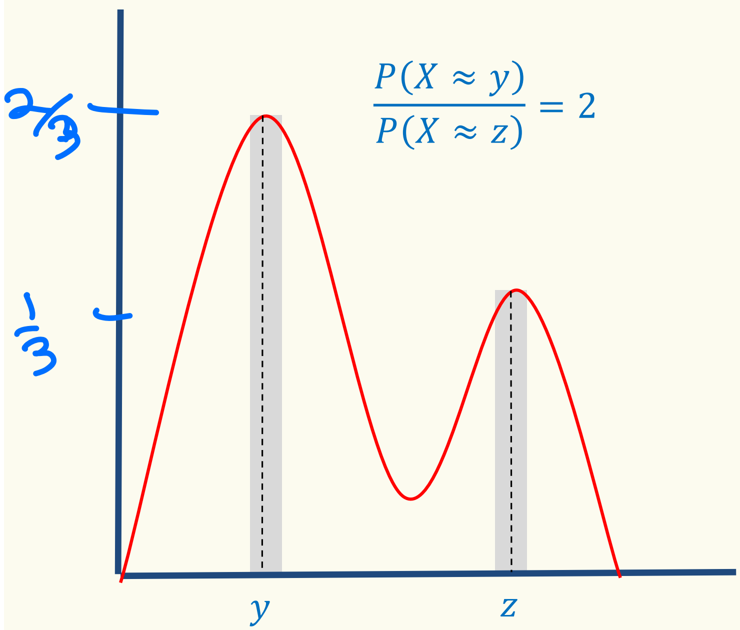
$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

What $f_X(x)$ measures: The local ***rate*** at which probability accumulates

Probability Density Function - Intuition



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Cumulative Distribution Function

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$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\, \mathrm{d}x$$

$$\| $$

$$P(X < a)$$

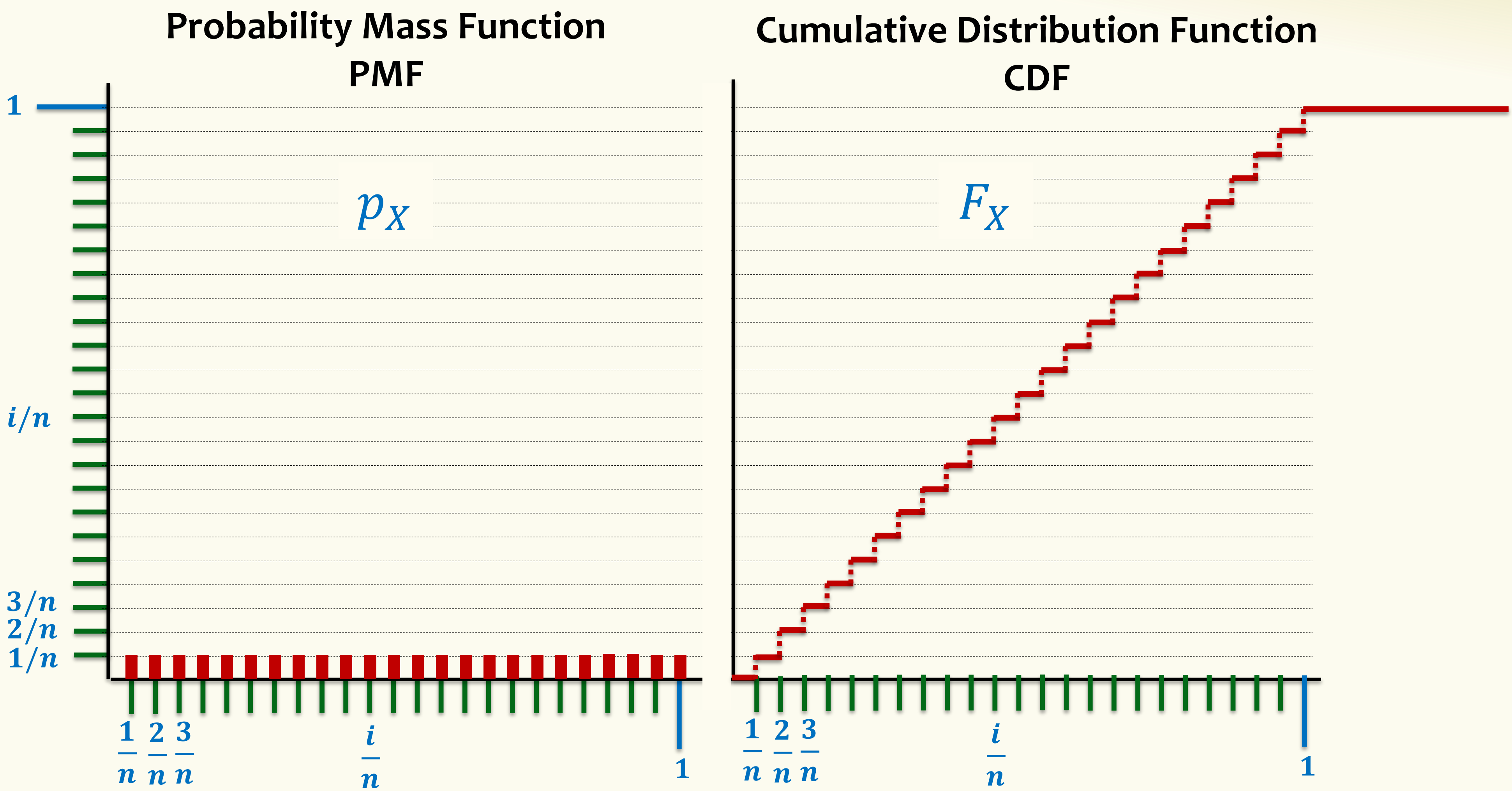
By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

A Discrete Approximation



$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

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PDF of Uniform RV

$X \sim \mathrm{Unif}(0,1)$

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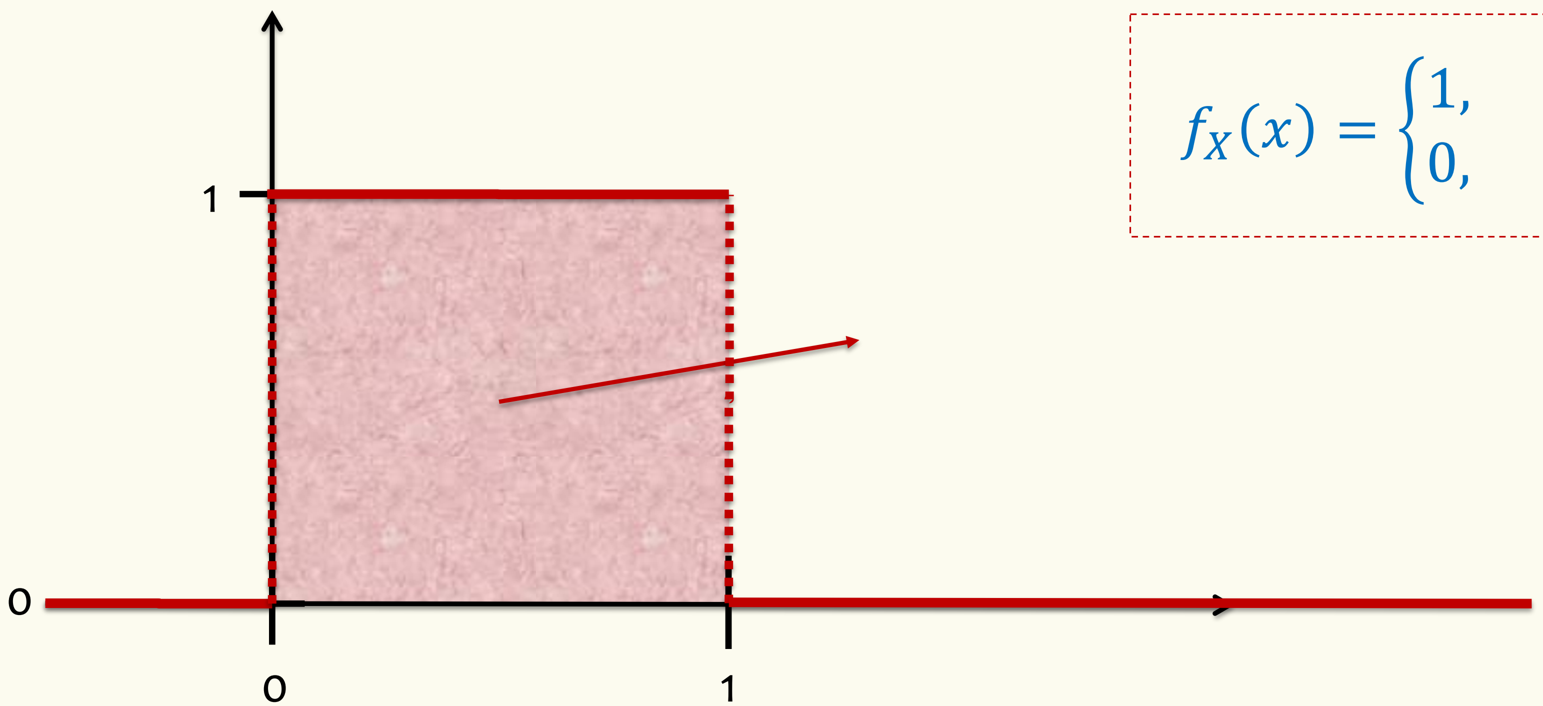
$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



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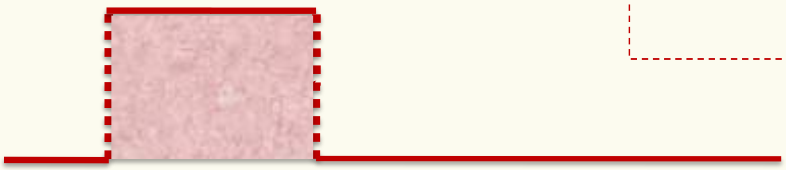

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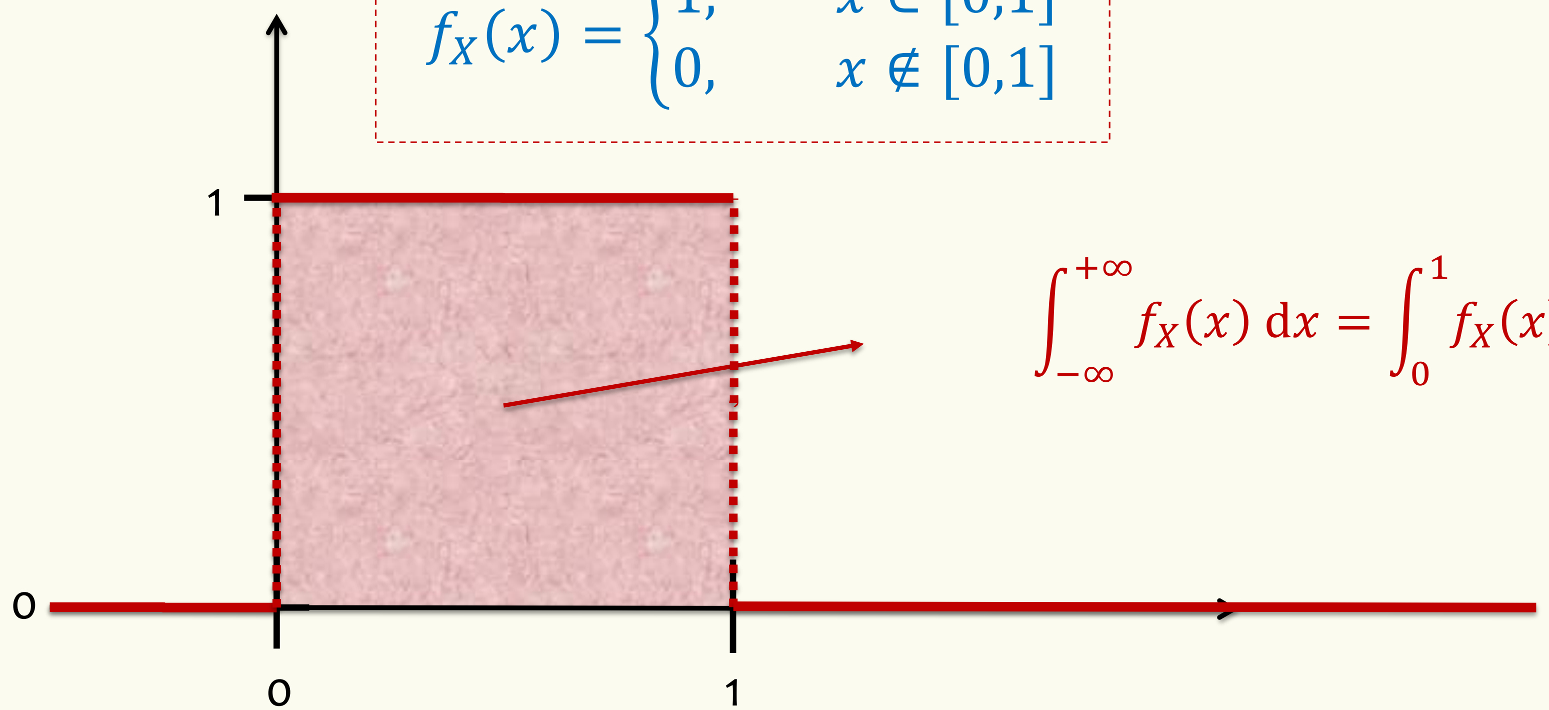
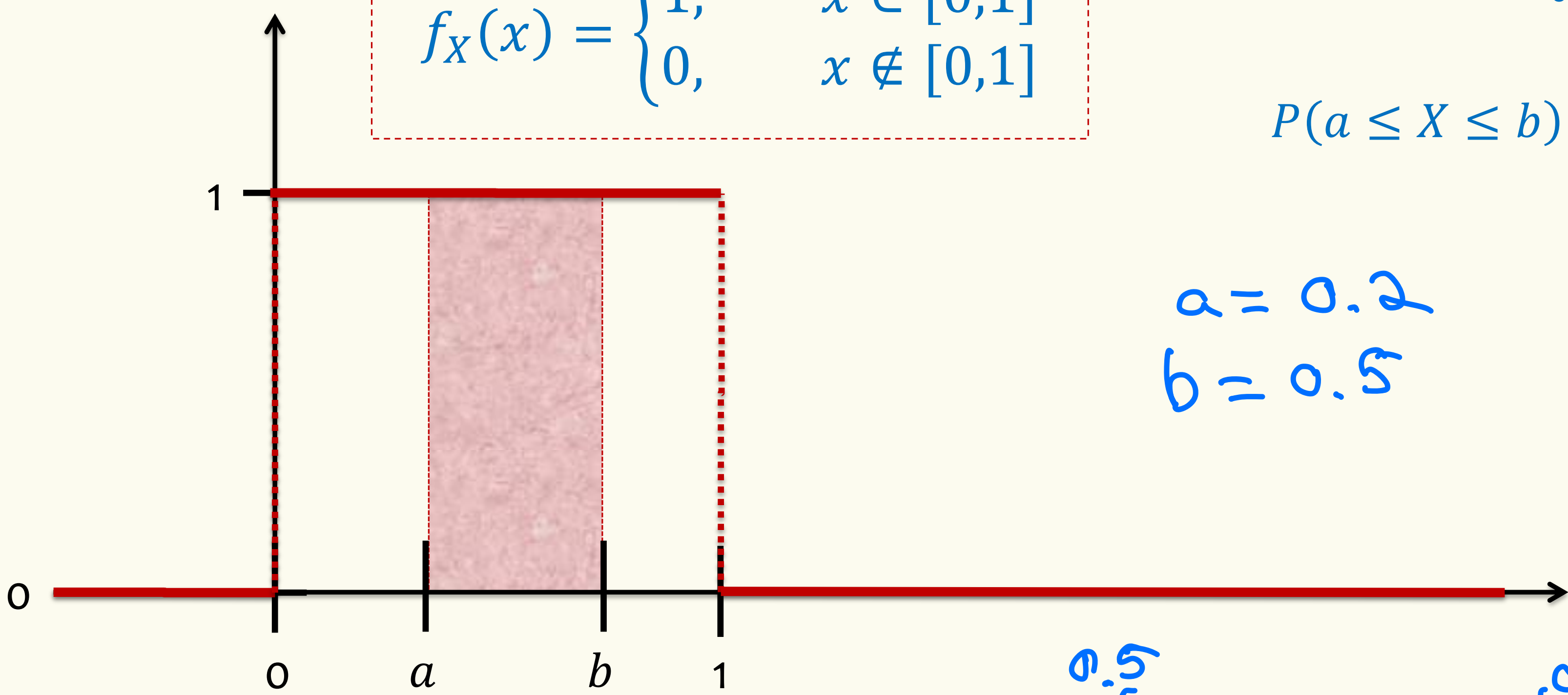


$$\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = \int_0^1 f_X(x)\, \mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$a = 0.2$
$b = 0.5$

$$\int_{0.2}^{0.5} 1\,dx = x\Big|_{0.2}^{0.5}$$

$= 0.3$

Probability of Event

$X \sim \mathrm{Unif}(0,1)$

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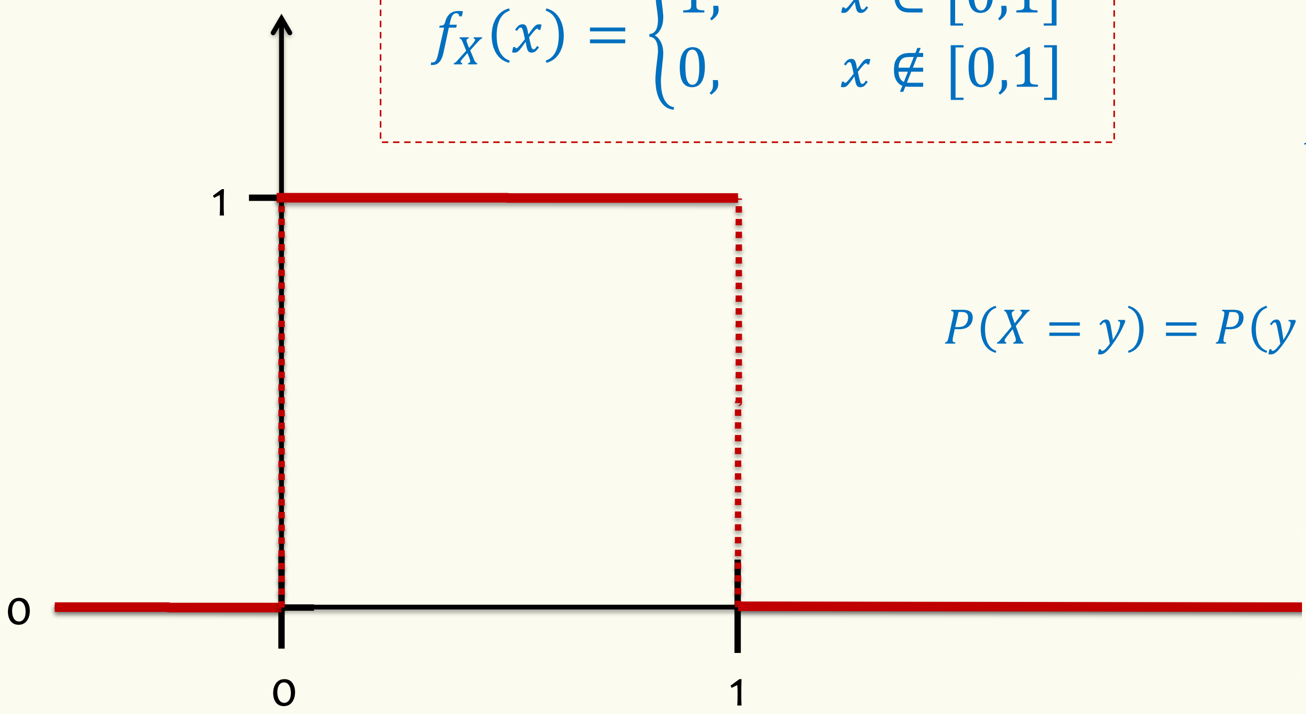

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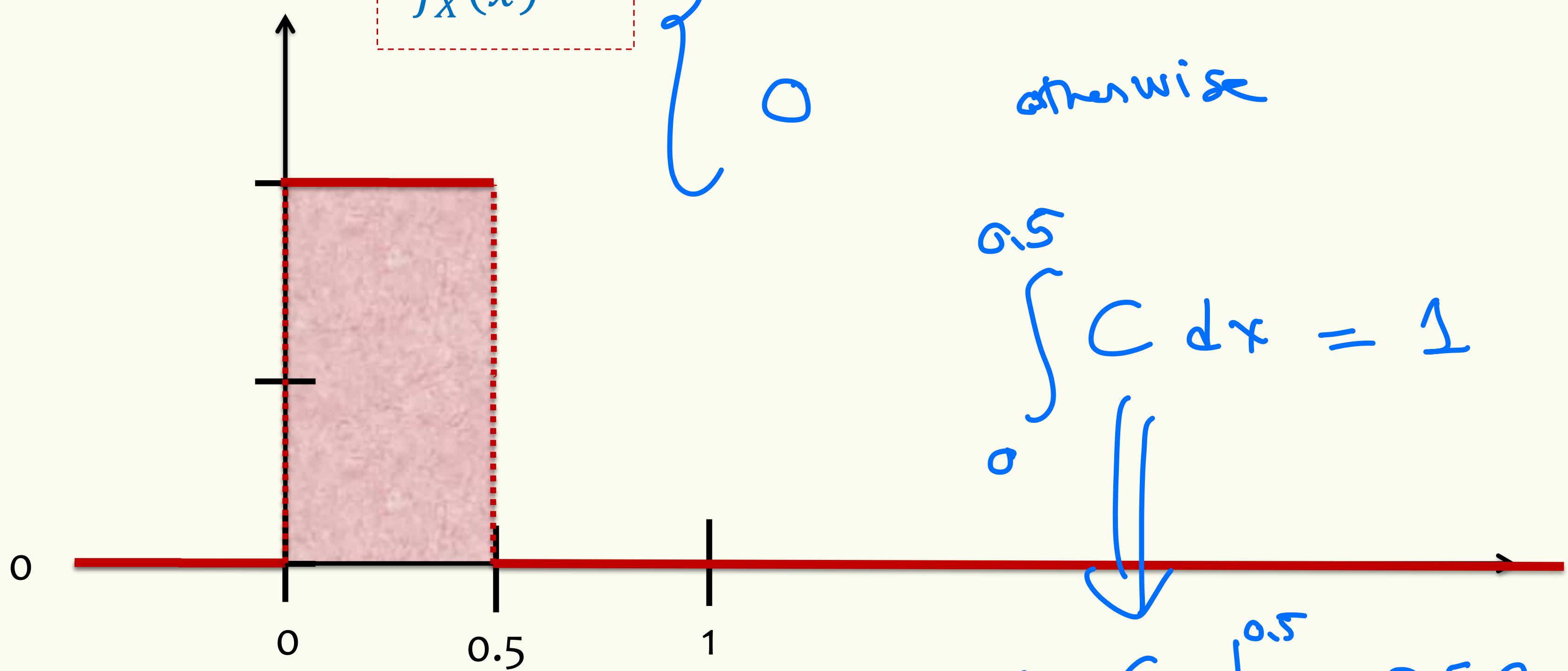
PDF of Uniform RV

$X \sim \text{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} C & 0 \le x \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$



$$\int_0^{0.5} C\,dx = 1$$

$$\Downarrow$$

$$= Cx\Big|_0^{0.5} = 0.5C$$

$$\Rightarrow C = 2!$$

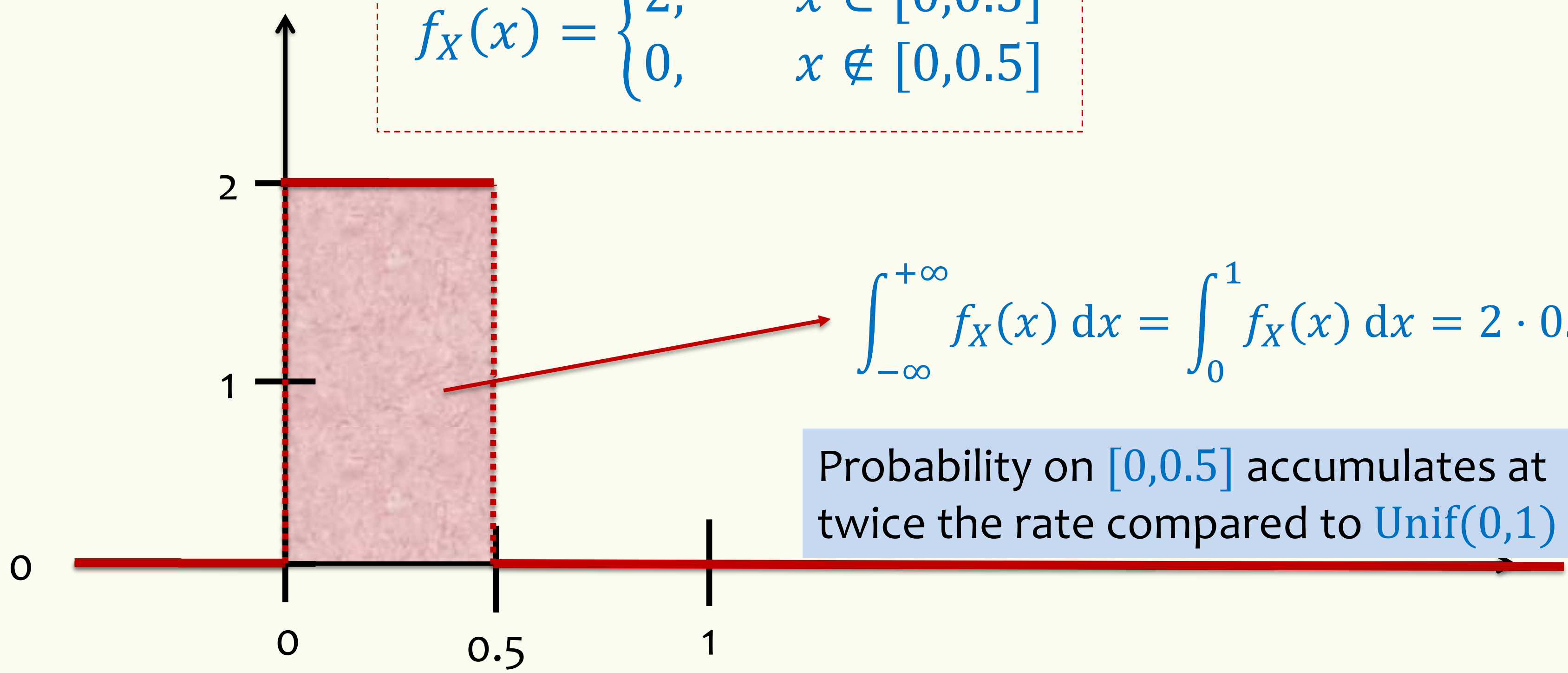
PDF of Uniform RV

$X \sim \mathrm{Unif}(0,0.5)$

Density $\neq$ Probability

$f_X(x) \gg 1$ is possible!

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Probability on $[0,0.5]$ accumulates at twice the rate compared to $\mathrm{Unif}(0,1)$

PDF of Uniform RV

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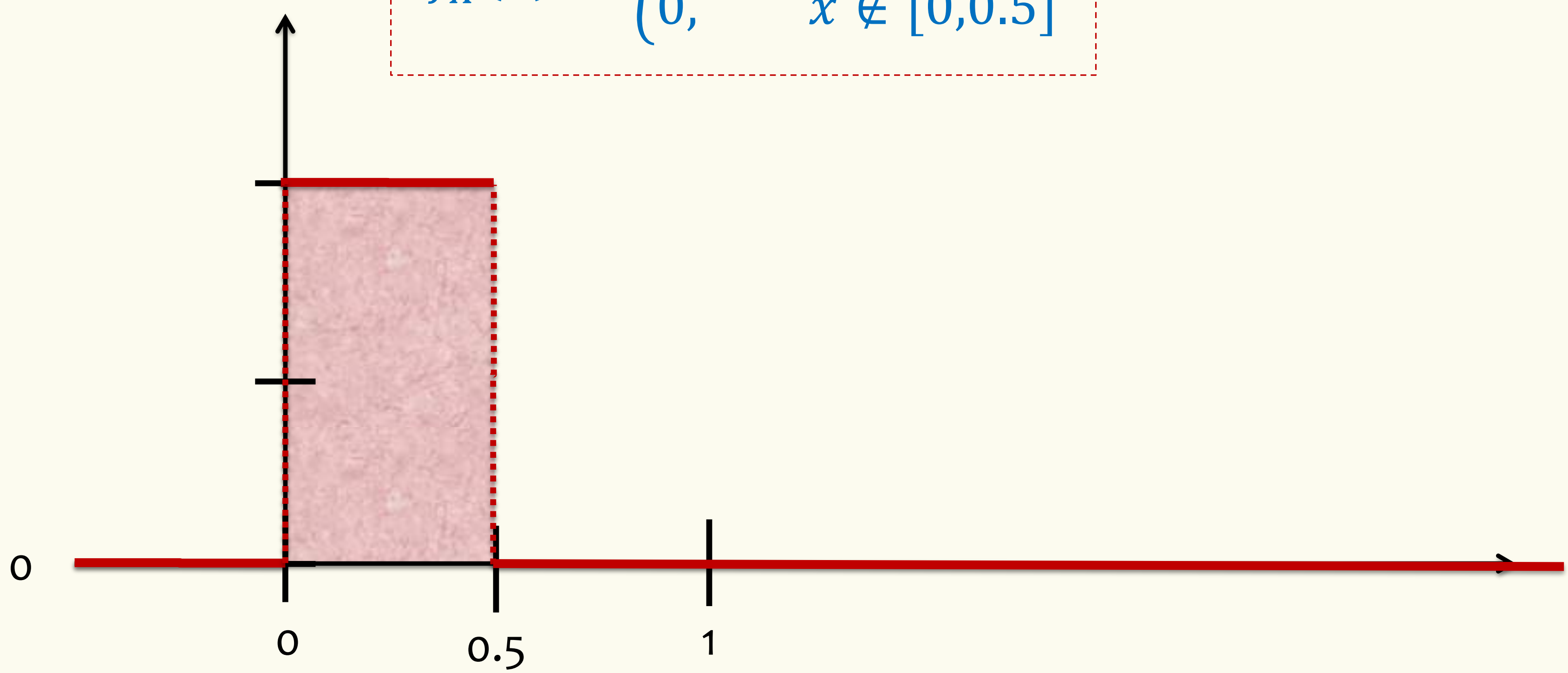
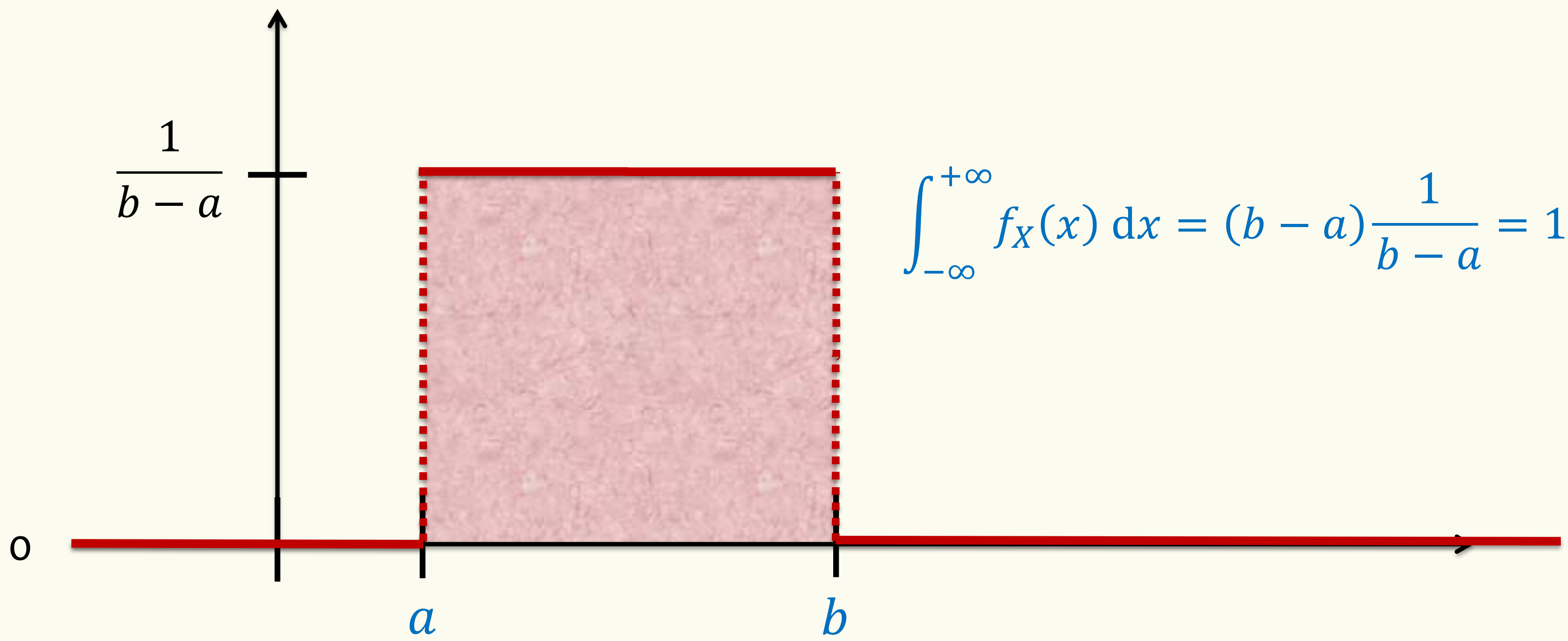


Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

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From Discrete to Continuous

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By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0$ $\quad$ $\lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$

Agenda

- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance of continuous r.v.
- Introduction to continuous zoo

Y discrete

$E(Y) = \sum_{x \in \Omega_Y} P_Y(x)\, x$

Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Proof follows same ideas as discrete case

$\mathrm{Var}(Y) = E((Y - E(Y))^2) = E(Y^2) - [E(Y)]^2$

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Proofs follow same ideas as discrete case

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, \mathrm{d}x = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$E[g(Y)] = \sum_{y \in \Omega_Y} g(y)\, p_Y(y)$ $\qquad$ $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

$g(Y) = Y$

$$\sum_y y^2 p_Y(y) \qquad \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

Agenda

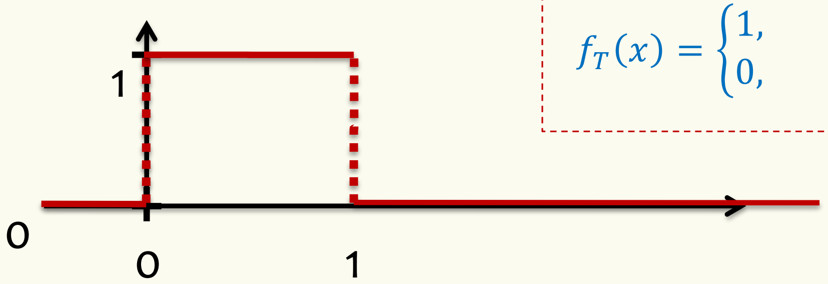
- Zoo of continuous random variables
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution

Expectation of a Continuous RV

Definition.

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Example. $T \sim \mathrm{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

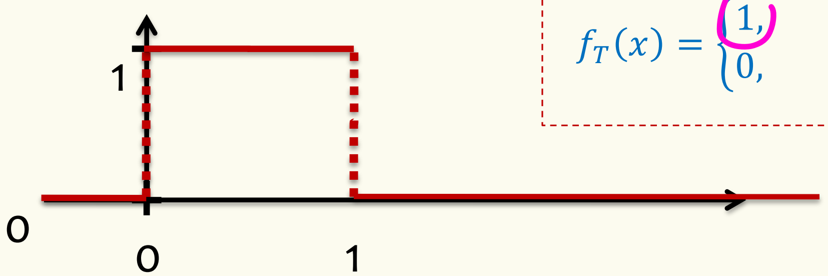
Expectation of a Continuous RV

Example. $T \sim \text{Unif}(0,1)$

Definition

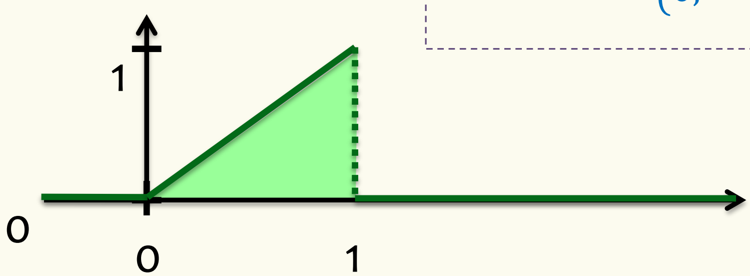
$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\mathbb{E}[X] = \int_0^1 1 \cdot x \, dx$$

$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\mathbb{E}[T] = \underbrace{\frac{1}{2} 1^2}_{\text{Area of triangle}} = \frac{1}{2}$$

Uniform Density – Expectation

$X \sim \mathrm{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x \quad = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} \quad = \frac{a+b}{2}$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a, b)$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2 \,\mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}[X] = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

Uniform Density – Variance

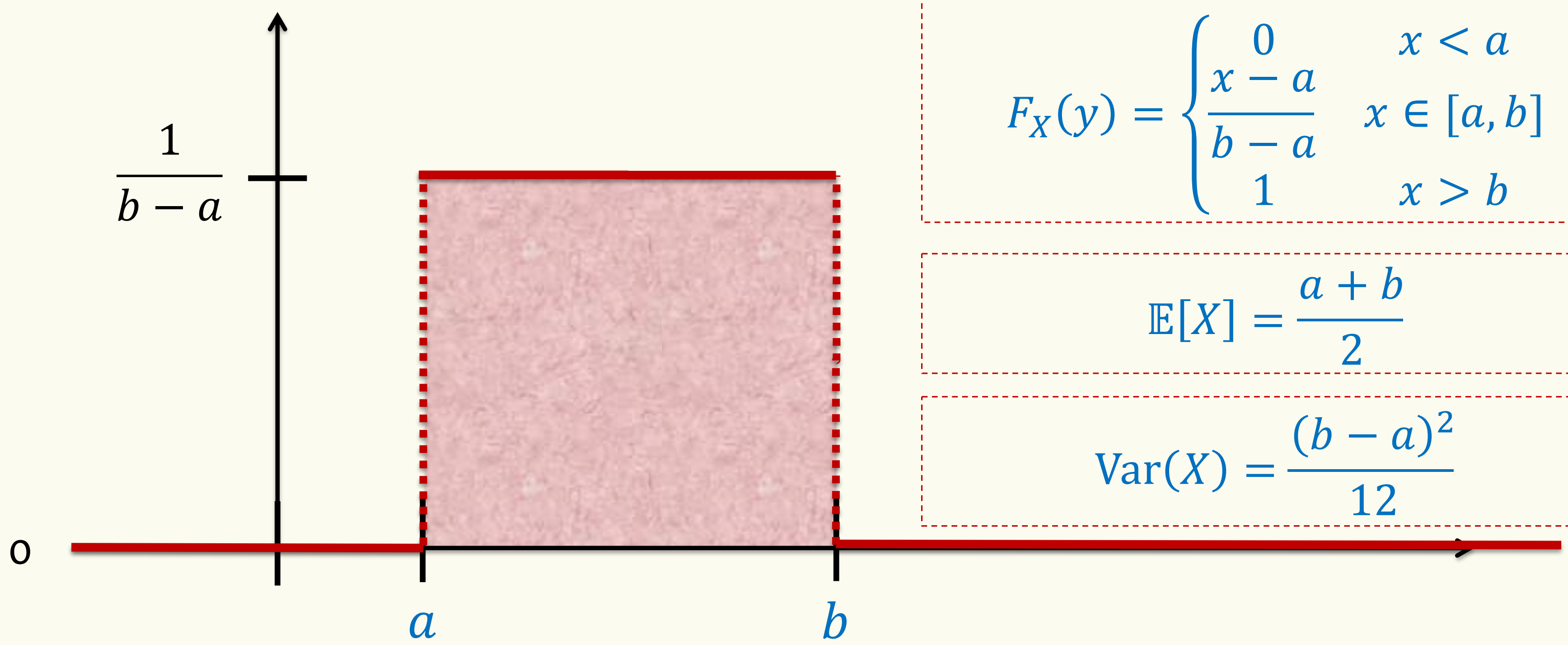
$$\mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}[X] = \frac{a+b}{2}$$

$X \sim \mathrm{Unif}(a, b)$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$

Uniform Distribution Summary

$X \sim \text{Unif}(a, b)$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a,b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Agenda

- Zoo of continuous random variables
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution

Exponential Density

Assume expected # of occurrences of an event per unit of time is λ (independently)

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER
- Rate of radioactive decay

Numbers of occurrences of event in one unit of time: Poisson distribution

$$P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!} \qquad \text{(Discrete)}$$

How long to wait until next event? Exponential density!

Let's define it and then derive it!

Exponential Density - Warmup

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!}$

Assume expected # of occurrences of an event per unit of time is λ (independently)

What is $\mathbb{E}[Z_t]$ where Z_t= # occurrences of event per t units of time?

$E(Z_1) = \lambda$ $\quad t = 1$

$E(Z_3) = 3\lambda$ $\quad t = 3$

$E(Z_{0.2}) = 0.2\lambda$ $\quad t = 0.2$

$Z_t \sim \text{Poisson}(\lambda t)$

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!}$

Exponential Density - Warmup

Assume expected # of occurrences of an event per unit of time is λ (independently)

What is the distribution of Z_t= # occurrences of event per t units of time?

$$\mathbb{E}[Z_t] = t\lambda$$

Z_t is independent over disjoint intervals

So $Z_t \sim Poi(t\lambda)$

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!}$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? <u>Exponential density!</u>

- Let X be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ is the # of events in the first t units of time, for $t \geq 0$.

$P(X > t) = P(Z_t = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!} = e^{-\lambda t}$

$F_X(t) = P(X \leq t) = 1 - e^{-\lambda t}$

$f_X(t) = \frac{d}{dt}F_X(t) = \lambda e^{-\lambda t} \quad t > 0$

$0 \quad t \leq 0$

$W \sim Poi(\lambda) \Rightarrow P(W = i) = e^{-\lambda}\frac{\lambda^i}{i!}$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ (independently)

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2, \dots\}$
- Let $X \sim Exp(\lambda)$ be the time till the first event. We will compute $F_X(t)$ and $f_X(t)$
- We know $Z_t \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $P(X > t) = P(\text{no event in the first } t \text{ units}) = P(Z_t = 0) = e^{-t\lambda}\frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_X(t) = P(X \leq t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_X(t) = \frac{d}{dt}F_X(t) = \lambda e^{-t\lambda}$

$P(X > t) = e^{-t\lambda}$

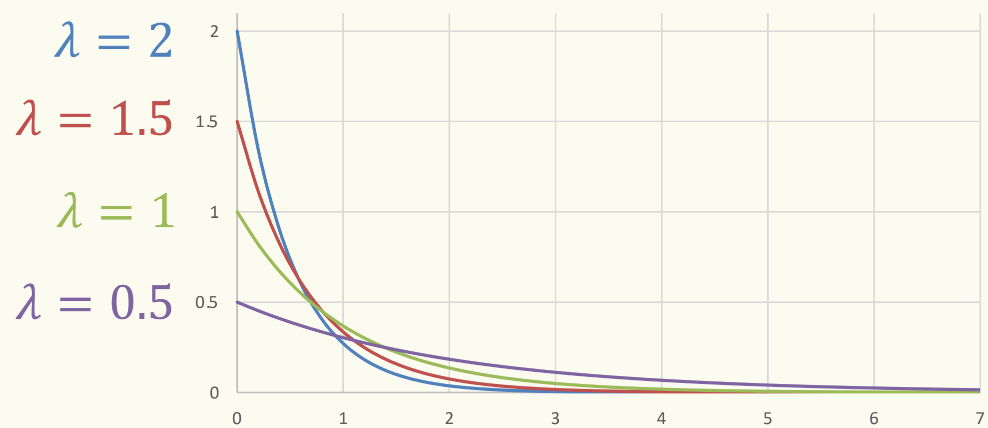
Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \, \mathrm{d}x$$

$$= \left(-(x + \frac{1}{\lambda}) e^{-\lambda x} \right) \Bigg|_0^{\infty} = \frac{1}{\lambda}$$

Somewhat complex calculation
use integral by parts

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Exponential Distribution

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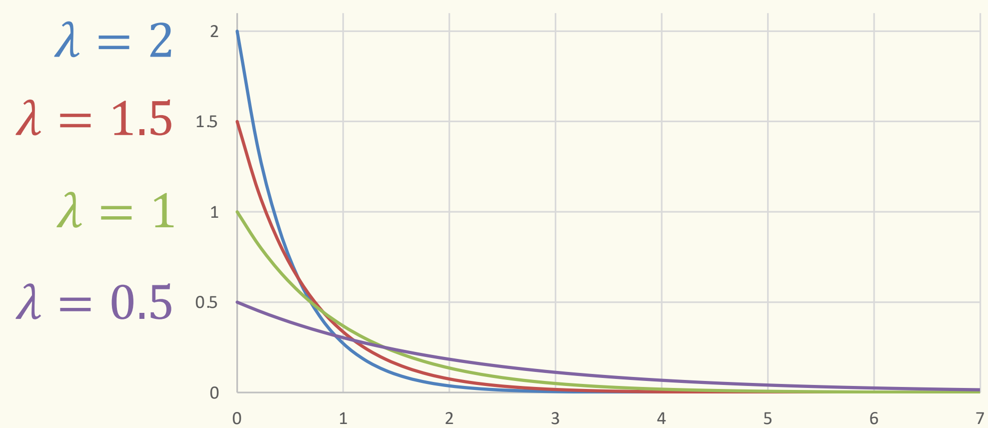

Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$P(X > s + t \mid X > s) = P(X > t).$$

Fact. $X \sim \mathrm{Exp}(\lambda)$ is memoryless.

Assuming an exponential distribution, if you've waited s minutes,
The probability of waiting t more is exactly same as when $s = 0$.

$P(X > t) = e^{-\lambda t}$

Memorylessness of Exponential

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$P(X > s + t \mid X > s) =$

Memorylessness of Exponential

$P(X > t) = e^{-\lambda t}$

Proof that assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as when $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$P(X > s + t \mid X > s) = \frac{P(\{X > s + t\} \cap \{X > s\})}{P(X > s)}$$

$$= \frac{P(X > s + t)}{P(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
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- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim Exp(\frac{1}{10})$$

$$P(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} \, dx$$

$y = \frac{x}{10}$ so $dy = \frac{dx}{10}$

$$P(10 \leq T \leq 20) = \int_{1}^{2} e^{-y} \, dy = -e^{-y} \Big|_{1}^{2} = e^{-1} - e^{-2}$$

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim Exp(\frac{1}{10})$$

so $F_T(t) = 1 - e^{-\frac{t}{10}}$

$$P(10 \le T \le 20) = F_T(20) - F_T(10)$$
$$= 1 - e^{-\frac{20}{10}} - \left(1 - e^{-\frac{10}{10}}\right) = e^{-1} - e^{-2}$$

Agenda

- Zoo
  - Uniform Distribution
  - Exponential Distribution
  - Normal Distribution

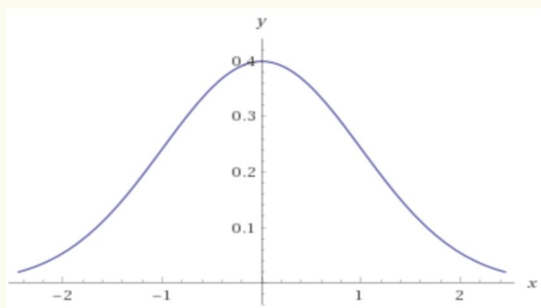
The Normal Distribution

Definition. A **Gaussian (or <u>normal</u>) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$.

Carl Friedrich Gauss



$\mathcal{N}(0, 1)$.

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Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}[X] = \mu$, and $\mathrm{Var}(X) = \sigma^2$

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The Normal Distribution

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Proof of expectation is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$, but proof for variance requires integration of $e^{-x^2/2}$

We will see next time why the normal distribution is (in some sense) the most important distribution.

The Normal Distribution

Aka a “Bell Curve” (imprecise name)