CSE 312

# Foundations of Computing II

## Lecture 13: Wrap up Poisson r.v.s + Bloom Filters

<u>Anna's office hours on Saturday (tmw) from 2-3pm</u>

# Agenda

- More on Poisson random variables
- An Application: Bloom Filters!

## Preview: Poisson

Model: $X$ is # events that occur in an hour

- Expect to see $3$ events per hour (but will be random)
- The expected number of events in $t$ hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

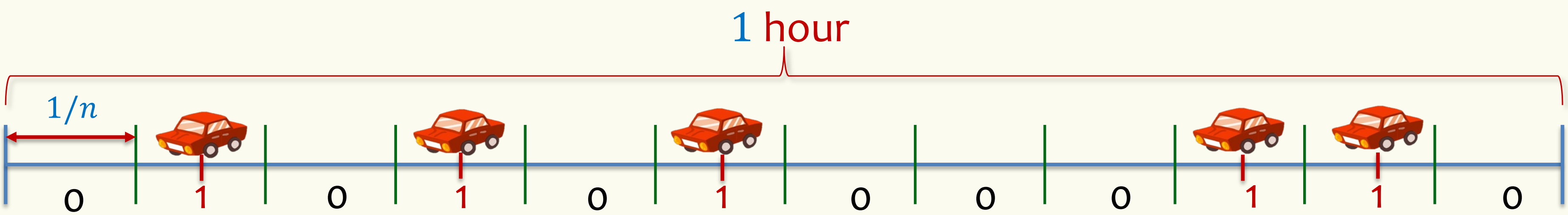
**Example – Modelling car arrivals at an intersection**

$X$ $=$ # of cars passing through a light in 1 hour

# Example – Model the process of cars passing through a light in 1 hour

$X$ = # cars passing through a light in $1$ hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$




**Discrete version:** $n$ intervals, each of length $1/n$ .

In each interval, there is a car with probability $p = \lambda/n$ (assume $\leq 1$ car can pass by)

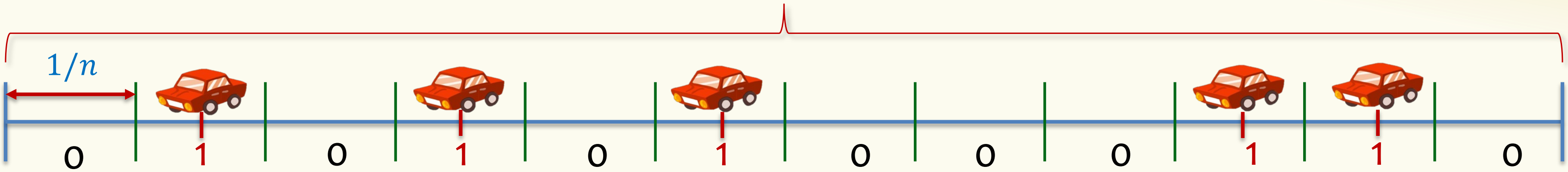
**Each interval is Bernoulli:** $X_i = 1$ if car in $i^{\text{th}}$ interval ($0$ otherwise). $P(X_i = 1) = \lambda\,/n$

$X = \sum_{i=1}^{n} X_i$ $\quad X \sim \text{Bin}(n, p)$ $\quad P(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}[X] = pn = \lambda$

# Don’t like discretization

$X$ is <u>binomial</u> $P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^{i}\left(1-\frac{\lambda}{n}\right)^{n-i}$




**We want now $n\to\infty$**

$$P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^{i}\left(1-\frac{\lambda}{n}\right)^{n-i}=\underbrace{\frac{n!}{(n-i)!\,n^{i}}}_{\to 1}\frac{\lambda^{i}}{i!}\underbrace{\left(1-\frac{\lambda}{n}\right)^{n}}_{\to e^{-\lambda}}\underbrace{\left(1-\frac{\lambda}{n}\right)^{-i}}_{\to 1}$$

$$\to P(X=i)=e^{-\lambda}\cdot\frac{\lambda^{i}}{i!}$$

## Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of $\lambda$ per unit time.
- Let $X$ be the *actual* number of events happening in a given time unit. Then $X$ is a *Poisson* r.v. *with parameter* $\lambda$ (denoted $X \sim \mathrm{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

Several examples of "Poisson processes":

- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

$$\boxed{\begin{gathered} E(X) = \lambda \\ \mathrm{Var}(X) = \lambda \end{gathered}}$$

# Poisson Random Variables

**Definition.** A **Poisson random variable** $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson approximates binomial when:

$n$ is very large, $p$ is very small, and $\lambda = np$ is "moderate"

e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

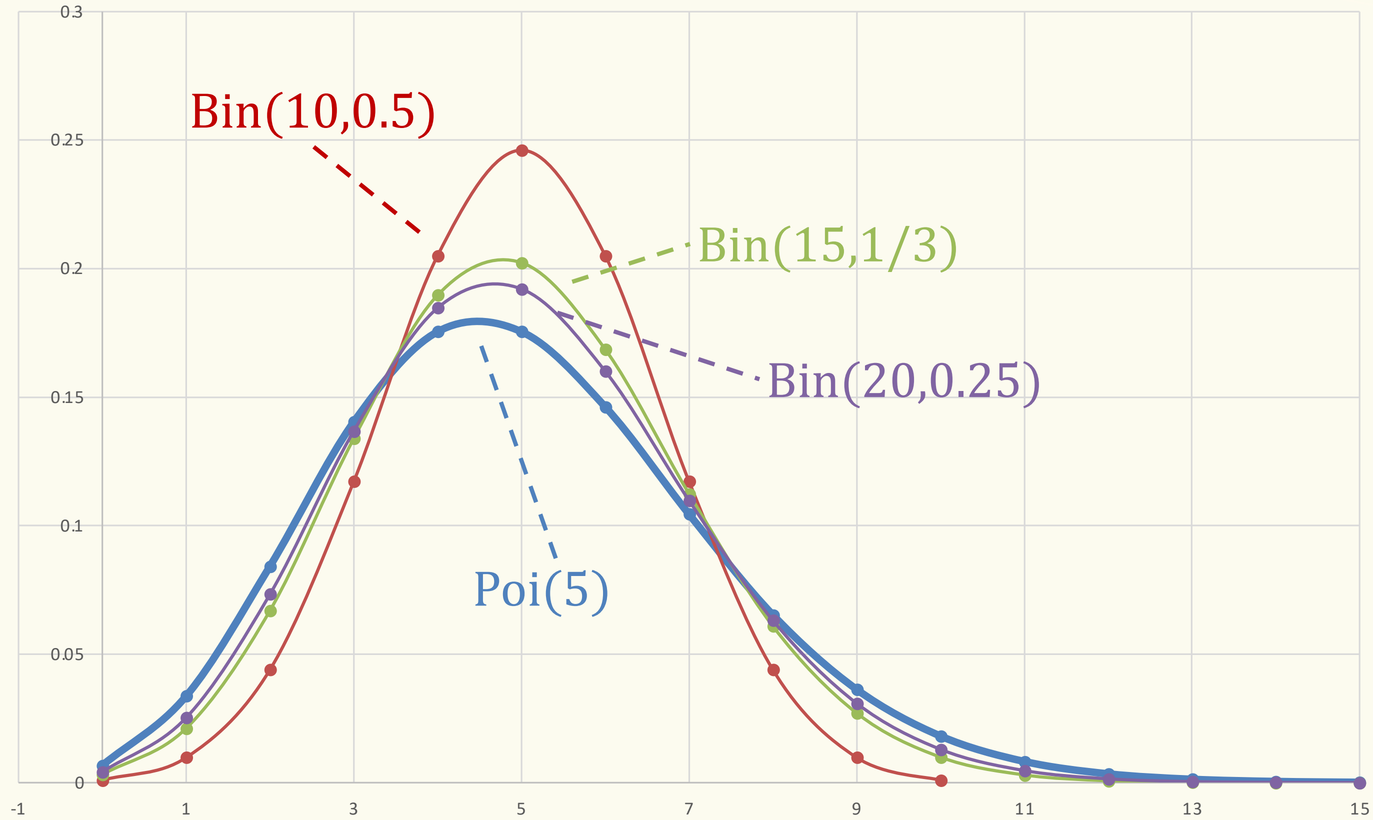
Formally, Binomial approaches Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

# Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10,15,20$




$$as\ n \to \infty,\ \ Binomial(n,\ \ p = \ \lambda/n) \to poi(\lambda)$$

## Sum of Independent Poisson RVs

Let $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. What kind of random variable is $Z$ ?

Aka what is the "distribution" of $Z$ ?

Intuition first:

- $X$ is measuring number of (type 1) events that happen in, say, an hour if they happen at an average rate of $\lambda_1$ per hour.
- $Y$ is measuring number of (type 2) events that happen in, say, an hour if they happen at an average rate of $\lambda_2$ per hour.
- $Z$ is measuring total number of events of both types that happen in, say, an hour, if type 1 and type 2 events occur independently.

## Sum of Independent Poisson RVs

**Theorem.** Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots,$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \cdots, X_n \sim \text{Poi}(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

**Theorem.** Let $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots,$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

## Proof

$P(Z = z) = \sum_{j=0}^{z} P(X = j, Y = z - j)$ Law of total probability

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$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$ Law of total probability

$= \Sigma_{j=0}^{z} P(X = j)\, P(Y = z - j) = \Sigma_{j=0}^{z}\ e^{-\lambda_1} \cdot \dfrac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \dfrac{\lambda_2^{z-j}}{z - j!}$ Independence

$= e^{-\lambda_1 - \lambda_2} \left( \Sigma_{j=0}^{z} \cdot \dfrac{1}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$

$= e^{-\lambda} \left( \Sigma_{j=0}^{z} \dfrac{z!}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \dfrac{1}{z!}$

$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$ Binomial Theorem

# Poisson Random Variables

**Definition.** A **Poisson random variable** $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**General principle:**

- Events happen at an average rate of $\lambda$ per time unit
- Number of events happening at a time unit $X$ is distributed according to Poi($\lambda$)
- Poisson approximates Binomial when $n$ is large, $p$ is small, and $np$ is moderate
- Sum of independent Poisson is still a Poisson

# Agenda

- Wrap up Poisson random variables
- An Application: Bloom Filters!

# Basic Problem

**Problem:** Store a subset $S$ of a large set $U$.

**Example.** $U =$ set of 128 bit strings $\quad |U| \approx 2^{128}$
$S =$ subset of strings of interest $\quad |S| \approx 1000$

**Two goals:**

1. **Very fast** (ideally constant time) answers to queries "Is $x \in S$?" for any $x \in U$.
2. **Minimal storage** requirements.

# Naïve Solution I – Constant Time

$$A[x] = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

**Idea:** Represent $S$ as an array $A$ with $2^{128}$ entries.

$S = \{0,2, \dots, K\}$

| 0 | 1 | 2 | … | $K$ | … | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | … | 0 | 0 |

**Membership test:** To check. $x \in S$ just check whether $A[x] = 1$.

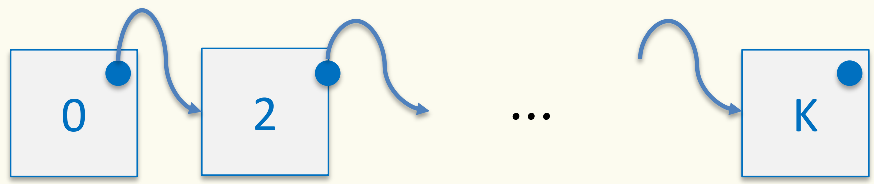
$\rightarrow$ **constant time!** 👍 😀

**Storage:** Require storing $2^{128}$ bits, even for small $S$. 👎 😢

## Naïve Solution II – Small Storage

**Idea:** Represent $S$ as a list with $|S|$ entries.

$S = \{0,2, \dots, K\}$




**Storage:** Grows with $|S|$ <u>only</u>

**Membership test:** Check $x \in S$ requires time linear in $|S|$

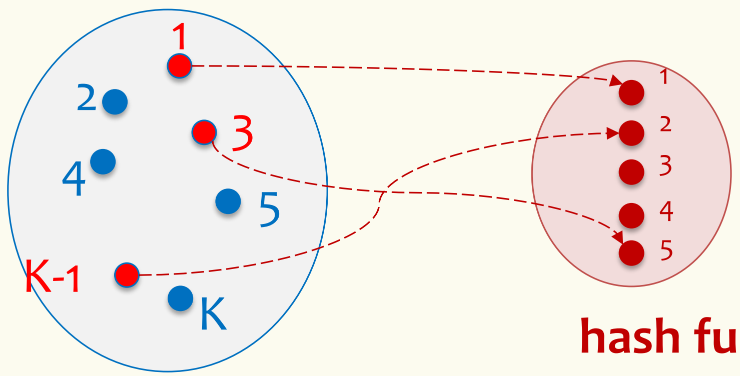
(Can be made logarithmic by using a tree)

## Hash Table

**Idea:** Map elements in $S$ into an array $A$ of size $m$ using a hash function $\mathbf{h}$

**Membership test:** To check $x \in S$ just check whether $A[\mathbf{h}(x)] = x$

**Storage:** $m$ elements (size of array)

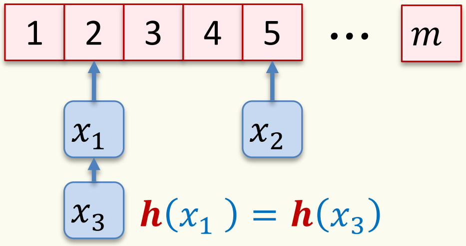



**hash function h**: $U \to [m]$

## Hashing: collisions

Collisions occur when $\boldsymbol{h}(x) = \boldsymbol{h}(y)$ for some distinct $x, y \in S$, i.e., two elements of set map to the same location

- Common solution: <u>chaining</u> – at each location (bucket) in the table, keep linked list of all elements that hash there.



# Hash Table

**Idea:** Map elements in $S$ into an array $A$ of size $m$ using a hash function $\mathbf{h}$

**Membership test:** To check $x \in S$ just check whether $A[\mathbf{h}(x)] = x$

**Storage:** $m$ elements (size of array)

**Challenge 1:** Ensure $\boldsymbol{h}(x) \neq \boldsymbol{h}(y)$ *for most* $x, y \in S$

**Challenge 2:** Ensure $m = O(|S|)$

## Good hash functions to keep collisions low

- The hash function $h$ is good iff it
  - distributes elements uniformly across the $m$ array locations so that
  - pairs of elements are mapped independently

"Universal Hash Functions" – see CSE 332

# Hashing: summary

**Hash Tables**

- They store the data itself
- With a good hash function, the data is well distributed in the table and lookup times are small.
- However, they need at least as much space as all the data being stored, i.e., $m = \Omega(|S|)$

In some cases, $|S|$ is huge, or not known a-priori …

Can we do better!?

# Bloom Filters

# to the rescue

(Named after Burton Howard Bloom)

## Bloom Filters – Main points

- Probabilistic data structure.
- Close cousins of hash tables.
  - But: Ridiculously space efficient
- Occasional errors, specifically false positives.

# Bloom Filters

- Stores information about a set of elements $S \subseteq U$.
- Supports two operations:
  1. **`add`**$(x)$ - adds $x \in U$ to the set $S$
  2. **`contains`**$(x)$ – ideally: true if $x \in S$, false otherwise

## Bloom Filters

- Stores information about a set of elements $S \subseteq U$.
- Supports two operations:
  1. **add**$(x)$ - adds $x \in U$ to the set $S$
  2. **contains**$(x)$ – ideally: true if $x \in S$, false otherwise

**Instead, relaxed guarantees:**
- False $\rightarrow$ **definitely** not in $S$
- True $\rightarrow$ **possibly** in $S$
  [i.e. we could have *false positives*]

## Bloom Filters – Why Accept False Positives?

- **Speed** – both **`add`** and **`contains`** very very fast.
- **Space** – requires a miniscule amount of space relative to storing all the actual items that have been added.
  - Often just 8 bits per inserted item!
- **Fallback mechanism** – can distinguish false positives from true positives with extra cost
  - Ok if mostly negatives expected + low false positive rate

## Bloom Filters: Application

- Google Chrome has a database of malicious URLs, but it takes a long time to query.
- Want an in-browser structure, so needs to be efficient and be space-efficient
- Want it so that can check if a URL is in structure:
  - If return False, then definitely not in the structure (don't need to do expensive database lookup, website is safe)
  - If return True, the URL may or may not be in the structure. Have to perform expensive lookup in this rare case.

# Bloom Filters – More Applications

- Any scenario where space and efficiency are important.
- Used a lot in networking
- Internet routers often use Bloom filters to track blocked IP addresses.
- In distributed systems when want to check consistency of data across different locations, might send a Bloom filter rather than the full set of data being stored.
- Google BigTable uses Bloom filters to reduce disk lookups
- And on and on…

## Bloom Filters – Ingredients

Basic data structure is a $k \times m$ binary array "the Bloom filter"

- $k$ rows $t_1, \ldots, t_k$, each of size $m$
- Think of each row as an $m$-bit vector

$k$ different hash functions $\mathbf{h}_1, \ldots, \mathbf{h}_k : U \to [m]$

# Bloom Filters - Initialization

Number of hash functions

Size of array associated to each hash function.

$$\boxed{\begin{array}{l}\textbf{function } \text{INITIALIZE}(k, m) \\ \quad \textbf{for } i = 1, \dots, k\textbf{: do} \\ \quad\quad t_i = \text{new bit vector of } m \text{ 0s}\end{array}}$$

for each hash function, initialize an empty bit vector of size $m$

## Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** INITIALIZE$(k, m)$
  **for** $i = 1, \ldots, k$: **do**
    $t_i$ = new bit vector of $m$ 0s

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 0 | 0 | 0 |
| $t_2$ | 0 | 0 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 0 |

## Bloom Filters: Add

**function** ADD$(x)$
    **for** $i = 1, \ldots, k$: **do**
        $t_i[h_i(x)] = 1$

for each hash function $\mathbf{h}_i$

Index into $i$-th bit-vector, at index produced by hash function and set to 1

$\mathbf{h}_i(x) \rightarrow$ result of hash function $\mathbf{h}_i$ on $x$

## Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{array}{l}\textbf{function } \text{ADD}(x)\\ \quad \textbf{for } i = 1, \ldots, k\textbf{: do}\\ \quad\quad t_i[h_i(x)] = 1\end{array}$$

add("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 0 | 0 | 0 |
| $t_2$ | 0 | 0 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 0 |

## Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\ &\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\ &\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

$h_2$("thisisavirus.com") → 1

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 0 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 0 |

## Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$
\begin{array}{l}
\textbf{function } \text{ADD}(x) \\
\quad \textbf{for } i = 1, \dots, k\textbf{: do} \\
\qquad t_i[h_i(x)] = 1
\end{array}
$$

add("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

$h_2$("thisisavirus.com") → 1

$h_3$("thisisavirus.com") → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
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Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\text{: } \textbf{do}\\&\qquad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

$h_2$("thisisavirus.com") → 1

$h_3$("thisisavirus.com") → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

## Bloom Filters: Contains

**function** CONTAINS$(x)$
    **return** $t_1[h_1(x)] == 1 \land t_2[h_2(x)] == 1 \land \cdots \land t_k[h_k(x)] == 1$

Returns True if the bit vector $t_i$ for each hash function has bit 1 at index determined by $h_i(x)$,
Returns False otherwise

## Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS($x$)
    **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

contains("thisisavirus.com")

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

# Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS($x$)
    **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True

contains("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

# Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS($x$)
  **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True True

contains("thisisavirus.com")

$h_1$("thisisavirus.com") → 2

$h_2$("thisisavirus.com") → 1

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
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# Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS($x$)
  **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True True True

contains(“thisisavirus.com”)

$h_1$(“thisisavirus.com”) → 2

$h_2$(“thisisavirus.com”) → 1

$h_3$(“thisisavirus.com”) → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

# Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS$(x)$
    **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

True    True    True

contains("thisisavirus.com")

$h_1$("thisisavirus.com") → 2
$h_2$("thisisavirus.com") → 1
$h_3$("thisisavirus.com") → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

Since all conditions satisfied, returns True (correctly)

## Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

add("totallynotsuspicious.com")

**function** ADD$(x)$
  **for** $i = 1, \ldots, k$: **do**
    $t_i[h_i(x)] = 1$

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 0 | 1 | 0 | 0 |
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## Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$
\begin{aligned}
&\textbf{function } \text{ADD}(x)\\
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&\quad\quad t_i[h_i(x)] = 1
\end{aligned}
$$

add("totallynotsuspicious.com")

$h_1$("totallynotsuspicious.com") $\rightarrow$ 1

| Index $\rightarrow$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
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# Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\textbf{function } \text{ADD}(x)$$
$$\quad \textbf{for } i = 1, \dots, k\textbf{: do}$$
$$\qquad t_i[h_i(x)] = 1$$

add("totallynotsuspicious.com")

$h_1$("totallynotsuspicious.com") → 1

$h_2$("totallynotsuspicious.com") → 0

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|---|---|---|---|---|---|
| $t_1$ | 0 | 1 | 1 | 0 | 0 |
| $t_2$ | 0 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

# Bloom Filters: False Positives

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\end{aligned}
$$

add("totallynotsuspicious.com")

$h_1$("totallynotsuspicious.com") → 1

$h_2$("totallynotsuspicious.com") → 0

$h_3$("totallynotsuspicious.com") → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
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$h_1$("totallynotsuspicious.com") → 1

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Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

**function** CONTAINS($x$)
  **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

contains("verynormalsite.com")

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 1 | 1 | 0 | 0 |
| $t_2$ | 1 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

# Bloom Filters: False Positives

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True

contains("verynormalsite.com")

$h_1$("verynormalsite.com") $\rightarrow$ 2

| Index $\rightarrow$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 1 | 1 | 0 | 0 |
| $t_2$ | 1 | 1 | 0 | 0 | 0 |
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True    True

contains("verynormalsite.com")

$h_1$("verynormalsite.com") → 2

$h_2$("verynormalsite.com") → 0

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 1 | 1 | 0 | 0 |
| $t_2$ | 1 | 1 | 0 | 0 | 0 |
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True    True    True

contains("verynormalsite.com")

$h_1$("verynormalsite.com") $\rightarrow$ 2
$h_2$("verynormalsite.com") $\rightarrow$ 0
$h_3$("verynormalsite.com") $\rightarrow$ 4

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| $t_1$ | 0 | 1 | 1 | 0 | 0 |
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# Bloom Filters: False Positives

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contains("verynormalsite.com")

$h_1$("verynormalsite.com") → 2

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$h_3$("verynormalsite.com") → 4

| Index → | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $t_1$ | 0 | 1 | 1 | 0 | 0 |
| $t_2$ | 1 | 1 | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | 0 | 0 | 1 |

Since all conditions satisfied, returns True (incorrectly)

## Bloom Filters – Three operations

- Set up Bloom filter for $S = \emptyset$

**function** INITIALIZE$(k, m)$
  **for** $i = 1, \ldots, k$: **do**
    $t_i$ = new bit vector of $m$ 0s

- Update Bloom filter for $S \leftarrow S \cup \{x\}$

**function** ADD$(x)$
  **for** $i = 1, \ldots, k$: **do**
    $t_i[h_i(x)] = 1$

- Check if $x \in S$

**function** CONTAINS$(x)$
  **return** $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

## What you can't do with Bloom filters

- There is no delete operation
  - Once you have added something to a Bloom filter for $S$, it stays
- You can't use a Bloom filter to name any element of $S$

But what you ***can*** do makes them very effective!


Brain Break

## Analysis: False positive probability

**Question:** For an element $x \in U$, what is the probability that `contains`$(x)$ returns true if `add`$(x)$ was never executed before?

## Analysis: False positive probability

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Probability over what?! Over the choice of the $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_k$

Assumptions for the analysis:

- Each $\mathbf{h}_i(x)$ is uniformly distributed in $[m]$ for all $x$ and $i$
- Hash function outputs for each $\mathbf{h}_i$ are mutually independent (not just in pairs)
- Different hash functions are independent of each other

# False positive probability – Events

Assume we perform $\mathbf{add}(x_1), \ldots, \mathbf{add}(x_n)$
+ $\mathbf{contains}(x)$ for $x \notin \{x_1, \ldots, x_n\}$

Event $E_i$ holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \ldots, \mathbf{h}_i(x_n)\}$

$$P(\text{false positive}) = P(E_1 \cap E_2 \cap \cdots \cap E_k) = \prod_{i=1}^{k} P(E_i)$$

$\mathbf{h}_1, \ldots, \mathbf{h}_k$ independent

## False positive probability – Events

Event $E_i$ holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \dots, \mathbf{h}_i(x_n)\}$

Event $E_i^c$ holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c) = \sum_{z=1}^{m} P(\mathbf{h}_i(x) = z) \cdot P(E_i^c \mid \mathbf{h}_i(x) = z)$$

**LTP**

## False positive probability – Events

Event $E_i^c$ holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c \mid \mathbf{h}_i(x) = z) = P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z \mid \mathbf{h}_i(x) = z)$$

Independence of values of $\boldsymbol{h}_i$ on different inputs

$$= P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z)$$

$$= \prod_{j=1}^{n} P(\mathbf{h}_i(x_j) \neq z)$$

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Independence of values of $\boldsymbol{h}_i$ on different inputs

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$$= \prod_{j=1}^{n} P(\mathbf{h}_i(x_j) \neq z)$$

Outputs of $\boldsymbol{h}_i$ uniformly spread

$$= \prod_{j=1}^{n} \left(1 - \frac{1}{m}\right) = \left(1 - \frac{1}{m}\right)^n$$

$$P(E_i^c) = \sum_{z=1}^{m} P(\mathbf{h}_i(x) = z) \cdot P(E_i^c \mid \mathbf{h}_i(x) = z) = \left(1 - \frac{1}{m}\right)^n$$

## False positive probability – Events

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$$P(E_i^c) = \left(1 - \frac{1}{m}\right)^n$$

$$\Rightarrow \quad \mathrm{FPR} = \prod_{i=1}^{k} \left(1 - P(E_i^c)\right) = \left(1 - \left(1 - \frac{1}{m}\right)^n\right)^k$$

# False Positivity Rate – Example

$$\text{FPR} = \left(1 - \left(1 - \frac{1}{m}\right)^{n}\right)^{k}$$

e.g., $n = 5{,}000{,}000$

$k = 30$

$m = 2{,}500{,}000$

FPR = 1.28%

## Comparison with Hash Tables - Space

- Google storing 5 million URLs, each URL 40 bytes.
- Bloom filter with $k = 30$ and $m = 2{,}500{,}000$

### Hash Table

(optimistic)
$5{,}000{,}000 \times 40B = 200\text{MB}$

### Bloom Filter

$2{,}500{,}000 \times 30 = 75{,}000{,}000$ bits

$< 10$ MB

## Time

- Say avg user visits 102,000 URLs in a year, of which 2,000 are malicious.
- 0.5 seconds to do lookup in the database, 1ms for lookup in Bloom filter.
- Suppose the false positive rate is 3%

0.5 seconds DB lookup

false positives

$$1\text{ms} + \frac{100000 \times 0.03 \times 500\text{ms} \quad + 2000 \times 500 \text{ ms}}{102000} \approx 25.51\text{ms}$$

total URLs

malicious URLs

Bloom filter lookup

## Bloom Filters typical of….

… randomized algorithms and randomized data structures.

- **Simple**
- **Fast**
- **Efficient**
- **Elegant**
- **Useful!**