CSE 312

Foundations of Computing II

Lecture 12: Zoo of Discrete RVs, continued

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Motivation for "Named" Random Variables

Random Variables that show up all over the place.

- Easily solve a problem by recognizing it's a special case of one of these random variables.

Each RV introduced today will show:

- A general situation it models
- Its name and parameters
- Its PMF, Expectation, and Variance
- Example scenarios you can use it

Agenda

- Zoo of Discrete RVs
  - Uniform Random Variables – last time
  - Bernoulli Random Variables – last time
  - Binomial Random Variables – last time
  - Geometric Random Variables
  - Poisson Random Variables

Welcome to the Zoo! (Preview) 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$
$$\mathbb{E}[X] = \frac{a+b}{2}$$
$$\text{Var}(X) = \frac{(b-a)(b-a+2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$
$$\mathbb{E}[X] = p$$
$$\text{Var}(X) = p(1-p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$\mathbb{E}[X] = np$$
$$\text{Var}(X) = np(1-p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1-p)^{k-1} p$$
$$\mathbb{E}[X] = \frac{1}{p}$$
$$\text{Var}(X) = \frac{1-p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$$
$$\mathbb{E}[X] = \frac{r}{p}$$
$$\text{Var}(X) = \frac{r(1-p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$
$$\mathbb{E}[X] = n\frac{K}{N}$$
$$\text{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$$

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \mathrm{Unif}(a, b)$

Range: $\Omega_X = \{a, a+1, \dots, b\}$

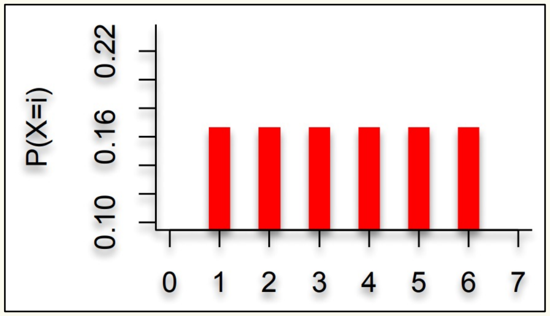
PMF: $\mathrm{P}(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

Range: $\Omega_X = \{0, 1\}$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Whether or not a share of a particular stock pays off or not
- Any indicator r.v.

Binomial Random Variables

A discrete random variable $X = \sum_{i=1}^{n} Y_i$ where each $Y_i \sim \mathrm{Ber}(p)$.

Counts number of successes in n **independent** trials, each with probability p of success.

X is a Binomial random variable

Notation: $X \sim \mathrm{Bin}(n, p)$

Range: $\Omega_X = \{0, 1, 2, \ldots, n\}$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\mathrm{Var}(X) = np(1-p)$

Examples:

- # of heads in n indep coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a particular bucket of a large hash table
- # of n different stocks that "pay off"

Mean, Variance of the Binomial

"i.i.d." is a commonly used phrase.
It means "independent & identically distributed"

If $Y_1, Y_2, \dots, Y_n \sim \text{Ber}(p)$ and independent (i.i.d.), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \text{Bin}(n, p)$

Claim $\mathbb{E}[X] = np$

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_1] = np$$

Claim $\text{Var}(X) = np(1-p)$

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \text{Var}(Y_i) = n\text{Var}(Y_1) = np(1-p)$$

Agenda

- Zoo of Discrete RVs
  - Recap
  - Geometric Random Variables
  - Poisson Random Variables

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ before seeing the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \mathrm{Geo}(p)$

Range:

PMF:

Expectation:

Variance:

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \text{Ber}(p)$ before seeing the first success.

X is called a Geometric random variable with parameter p.

Notation: $X \sim \text{Geo}(p)$

Range: $\Omega_X = \{1, 2, 3, \dots.\}$

PMF: $P(X = k) = (1 - p)^{k-1} p$

Expectation: $\mathbb{E}[X] = \frac{1}{p}$

Variance: $\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it
- # hash trials until a miner successfully mines a Bitcoin

Agenda

- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random Variables
  - More examples

Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What kind of random variable is this and what is $\mathbb{E}[X]$?

Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What kind of random variable is this and what is $\mathbb{E}[X]$?

Binomial (1024, 0.001)

Therefore $\mathbb{E}[X] = np = 1024 \cdot 0.001 = 1.024$

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What kind of random variable is this and what is $\mathbb{E}[X]$?

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What kind of random variable is this and what is $\mathbb{E}[X]$?

Probability that you play whole song without a mistake is 0.999^{1000}

Therefore X is a Geometric random variable with parameter $p = 0.999^{1000}$

So its expectation is $\frac{1}{0.999^{1000}}$

Agenda

- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Geometric Random variables
  - Examples
  - Poisson Distribution
    - Approximate Binomial distribution using Poisson distribution

Preview: Poisson

Model: X is # events that occur in an hour

- Expect to see 3 events per hour (but will be random)
- The expected number of events in t hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

Example – Modelling car arrivals at an intersection

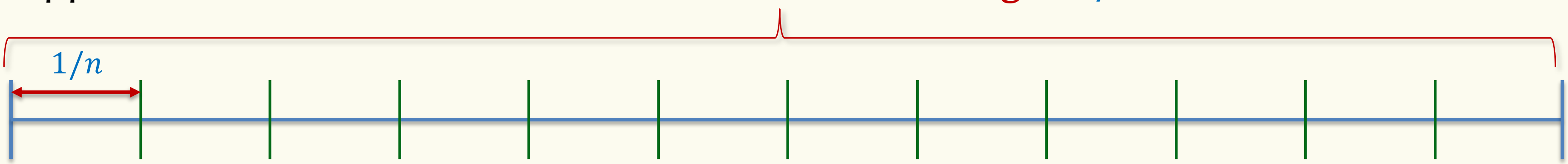
X $=$ # of cars passing through a light in 1 hour

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into n intervals of length $1/n$



This gives us n independent intervals

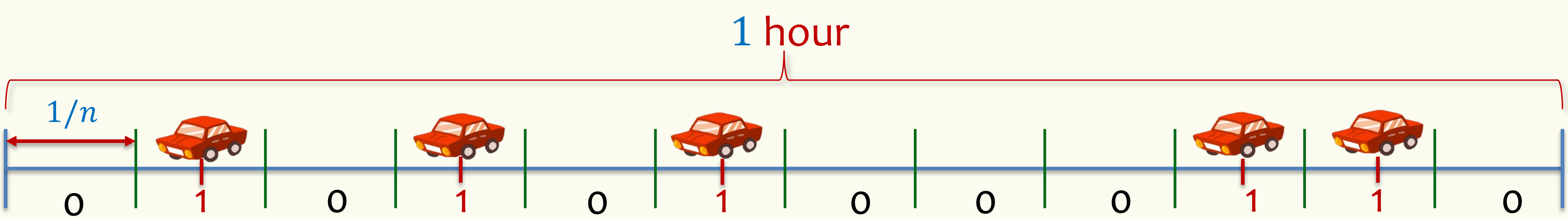
Assume either zero or one car per interval

$p =$ probability car arrives in a single interval of length $1/n$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = 3$



This gives us n independent intervals

Assume either zero or one car per interval

$p =$ probability car arrives in an interval

Model as $\text{Bin}(n, p)$

What should p be?

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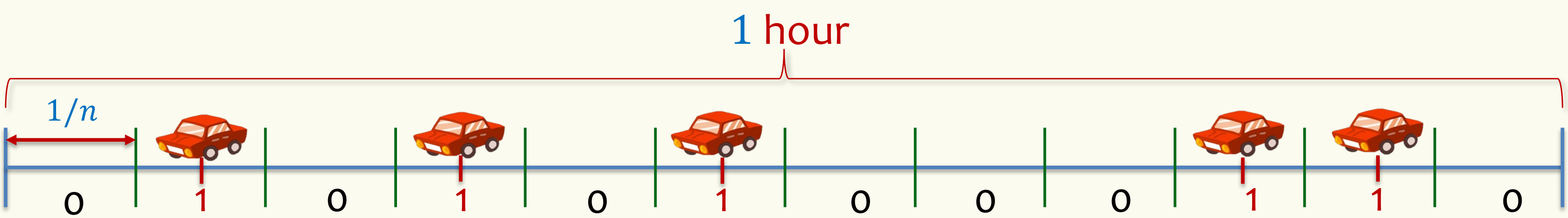
A. $3/n$
B. $3n$
C. 3
D. $3/60$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

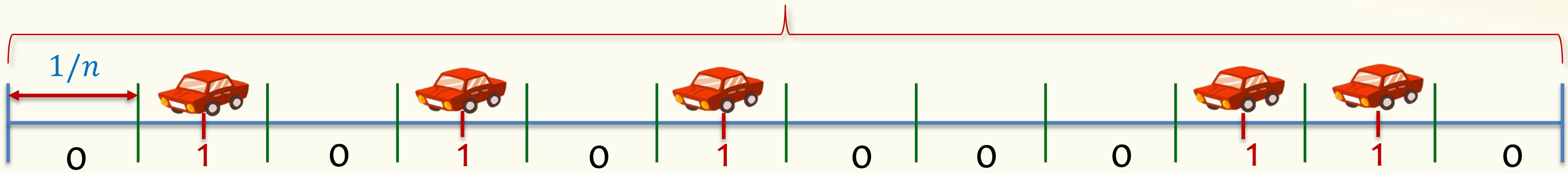
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n$

$X = \sum_{i=1}^{n} X_i$ $\quad X \sim \text{Bin}(n, p)$ $\quad P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don’t like discretization

X is <u>binomial</u> $P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$

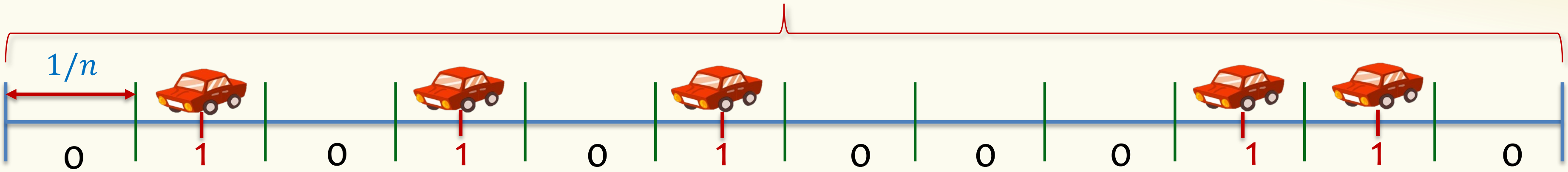


We want now $n \to \infty$

$$P(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$$

Don’t like discretization

X is binomial $P(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1 - \frac{\lambda}{n}\right)^{n-i}$



We want now $n \to \infty$

$$P(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1 - \frac{\lambda}{n}\right)^{n-i} = \underbrace{\frac{n!}{(n-i)!\,n^i}}_{\to 1}\frac{\lambda^i}{i!}\underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}}\underbrace{\left(1 - \frac{\lambda}{n}\right)^{-i}}_{\to 1}$$

$$\to P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted $X \sim \mathrm{Poi}(\lambda)$) and has distribution (PMF):

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

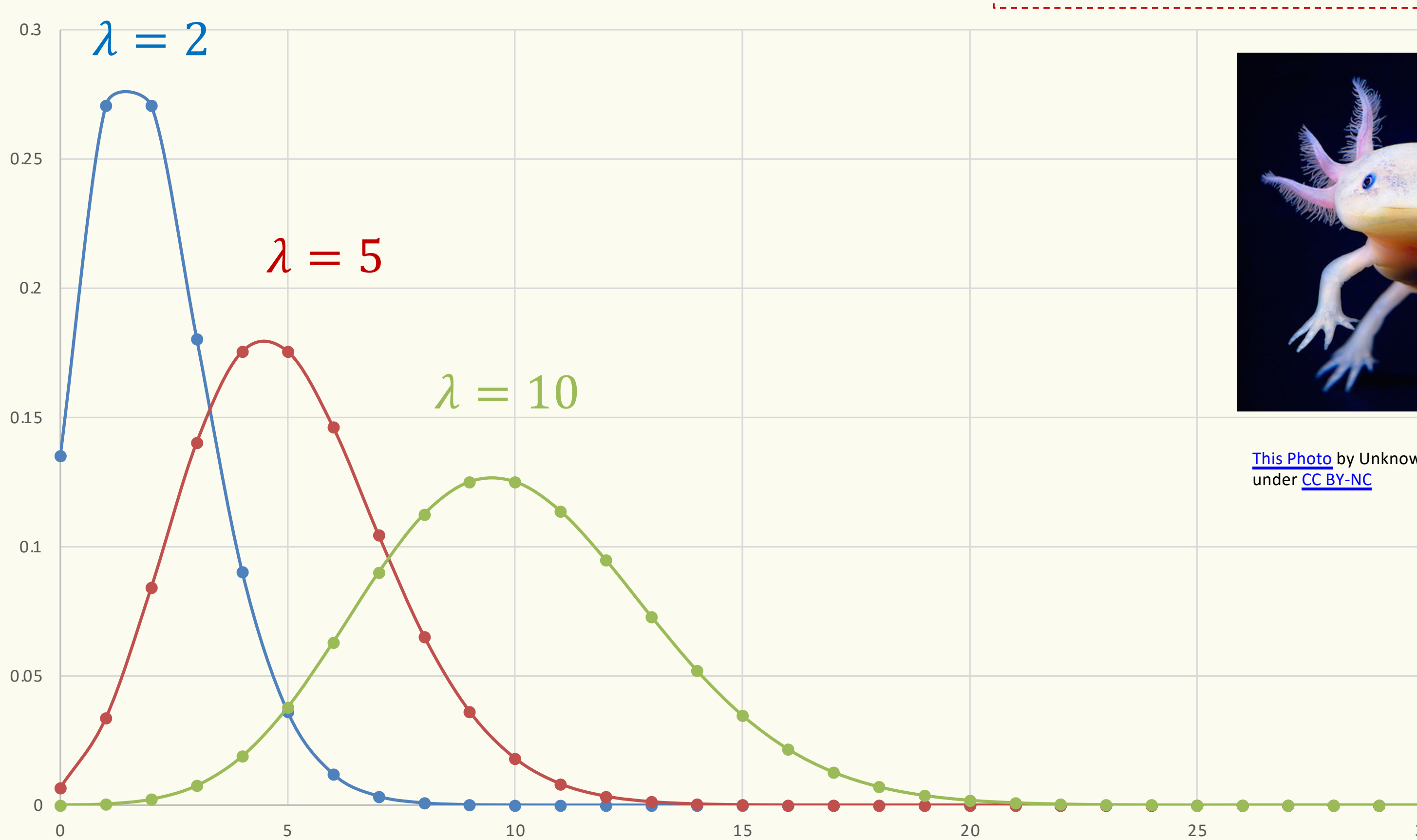
Several examples of "Poisson processes":

- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!} \qquad i = 0, 1, 2, \ldots$$

Is this a valid probability mass function?
(How do you show that a pmf is valid?)

Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

To show that a pmf is valid, need to check that it takes nonnegative values and that the probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}$$

Fact (Taylor series expansion):

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter $\lambda \geq 0$, then

$$\mathbb{E}[X] = ?$$

Proof. $$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i$$

Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}[X] = \lambda$$

Proof.

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

= 1 (see prior slides!)

Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $$\mathbb{E}[X^2] = \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

→ $$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Agenda

- Zoo of Discrete RVs
  - Uniform Random Variables, Part I
  - Bernoulli Random Variables, Part I
  - Binomial Random Variables, Part I
  - Poisson Distribution
    - Approximate Binomial distribution using Poisson distribution
  - Applications
  - Negative Binomial Random Variables
  - Hypergeometric Random Variables

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson approximates binomial when:

n is very large, p is very small, and $\lambda = np$ is "moderate"

e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

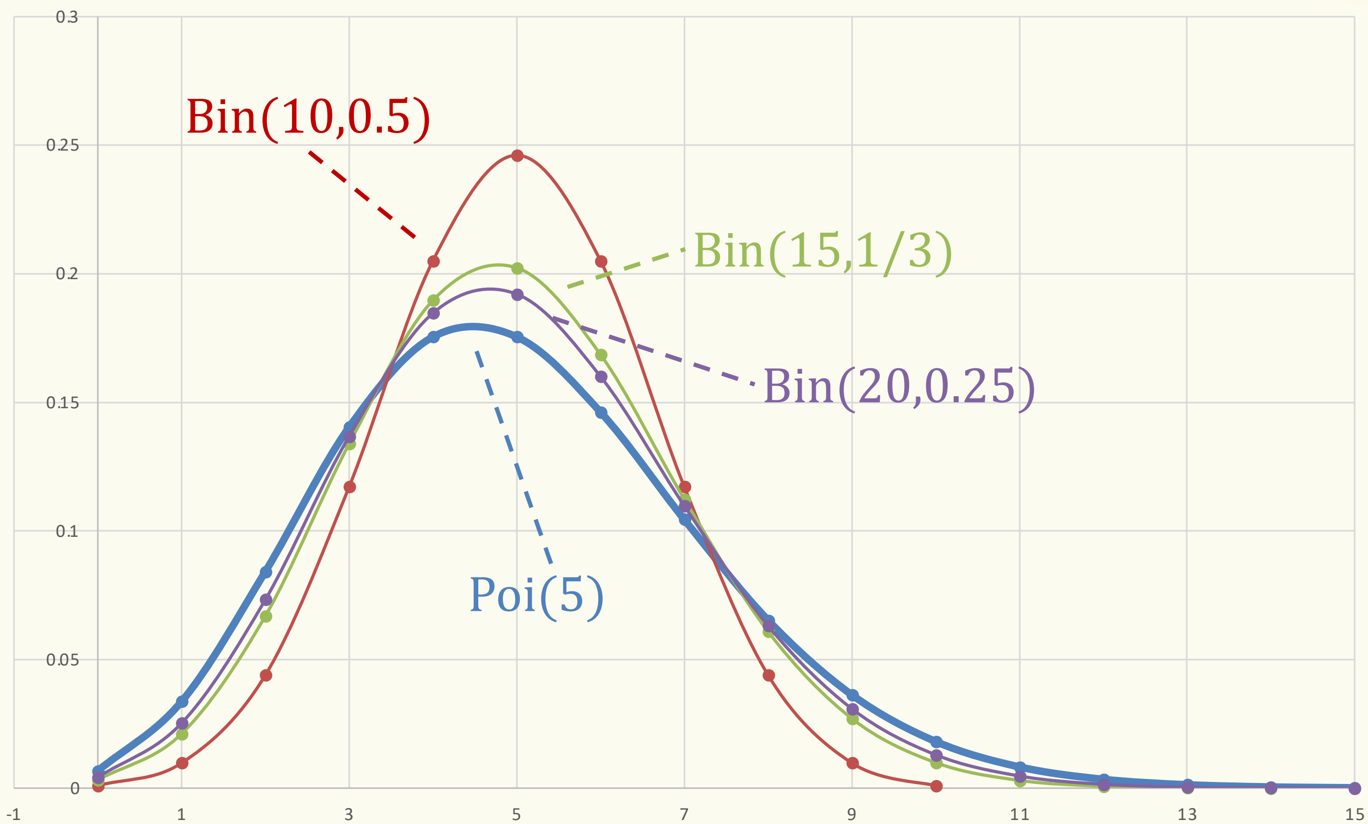
Formally, Binomial approaches Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10{,}15{,}20$



$as\ n \to \infty,\ \ Binomial(n,\ \ p = \lambda/n) \to poi(\lambda)$

From Binomial to Poisson

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1-p)$$

$$n \to \infty$$

$$np = \lambda$$

$$p = \frac{\lambda}{n} \to 0$$

$X \sim \text{Poi}(\lambda)$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$

What is probability that message arrives uncorrupted?

Using $X \sim \mathrm{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01)$

$$P(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} \approx 0.990049834$$

Using $Y \sim \mathrm{Bin}(10^4, 10^{-6})$

$$P(Y = 0) \approx 0.990049829$$

Sum of Independent Poisson RVs

Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. What kind of random variable is Z ?

Aka what is the "distribution" of Z ?

Intuition first:

- X is measuring number of (type 1) events that happen in, say, an hour if they happen at an average rate of λ_1 per hour.
- Y is measuring number of (type 2) events that happen in, say, an hour if they happen at an average rate of λ_2 per hour.
- Z is measuring total number of events of both types that happen in, say, an hour, if type 1 and type 2 events occur independently.

Sum of Independent Poisson RVs

Theorem. Let $X \sim \mathrm{Poi}(\lambda_1)$ and $Y \sim \mathrm{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots,$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

More generally, let $X_1 \sim \mathrm{Poi}(\lambda_1), \cdots, X_n \sim \mathrm{Poi}(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = X + Y$. For all $z = 0,1,2,3 \ldots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Proof

$P(Z = z) = \sum_{j=0}^{z} P(X = j, Y = z - j)$ Law of total probability

Proof

$$P(Z = z) = \Sigma_{j=0}^{z} P(X = j, Y = z - j)$$ Law of total probability

$$= \Sigma_{j=0}^{z} P(X = j)\, P(Y = z - j) = \Sigma_{j=0}^{z}\ e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z - j!}$$ Independence

$$= e^{-\lambda_1 - \lambda_2} \left(\Sigma_{j=0}^{z}\ \cdot \frac{1}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\Sigma_{j=0}^{z}\ \frac{z!}{j!\, z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$ Binomial Theorem

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)
- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson