



01 - Reminders!



PSet 2 grades released

(regrade requests open ~24 hours after grades are released and close after a week)

PSet 3

(written & coding part due yesterday)

PSet 4 is released! Start early :D





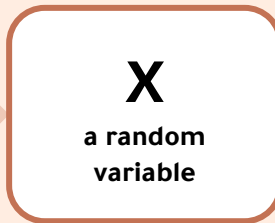
Review



02 - Random Variables



An outcome
from a random
experiment



Some number
(the **range of X** is the set of
possible values X can take on)



Probability Mass Function (PMF)

$P(X=k)$

probability that the random variable **X** will take on
the value **k**

what is the probability of an outcome that will result in X being k

for discrete random variables (random variables with a finite, countably infinite range), this may sometimes be a piecewise function



Random Variables



01

02

03

04



Random variable

Captures a quantitative property
(some numerical value that describes the
outcome) of the outcome in a random
experiment

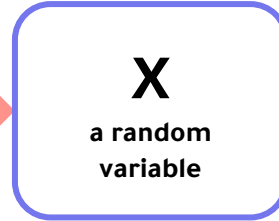
*e.g., sum of the dices on an random
experiment where we roll 2 dice*



Random Variables



An outcome
from a random
experiment



Some number
(the **range (or support)** of **X**
(sometimes denoted as Ω_X) is
the set of possible values **X** can
take on)

Probability Mass Function (PMF)

$$p_X(k) = P(X=k)$$

probability that the random
variable **X** will take on the value
k

*what is the probability of an outcome
that will result in **X** being **k***

for discrete random variables (random variables with a finite, countably
infinite range), this may sometimes be a piecewise function



01

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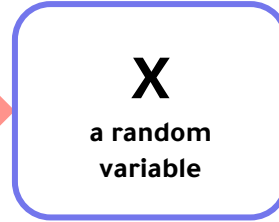




Random Variables



An outcome
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experiment



Some number
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take on)

Cumulative Distribution Function

$$F_X(k) = P(X \leq k) \rightarrow \text{probability that the value } X \text{ takes on is less than or equal to } k$$

what is the probability of an outcome that will result in X being $\leq k$

often can be derived from the PDF



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Random Variables



01

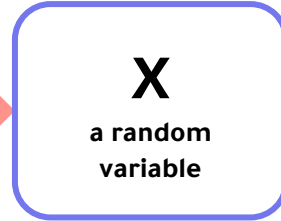
02

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04



An outcome
from a random
experiment



Some number
(the **range (or support)** of **X**
(sometimes denoted as Ω_X) is
the set of possible values **X** can
take on)

Expectation

$$E[X] = \sum(k \cdot P(X=k))$$

sum of values in the range of **X**,
weighted by the probability
*on average, what value can we "expect" **X** to take?*

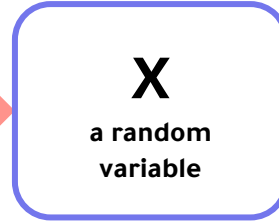
think about it like a weighted average of all the possible values **X** could be (weighted by the $P(X=k)$)



Random Variables



An outcome
from a random
experiment



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Expectation

$$E[X] = \sum(k \cdot P(X=k))$$

sum of values in the range of **X**,
weighted by the probability
*on average, what value can we "expect" **X** to take?*

just averaging all the possible values of **X** wouldn't work since each outcome isn't necessarily equally likely



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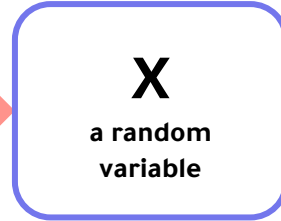




Random Variables



An outcome
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experiment



Some number
(the **range (or support)** of **X**
(sometimes denoted as Ω_X) is
the set of possible values **X** can
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Expectation of a function of **X** (aka “Law of the Unconscious Statistician”
(aka “LOTUS”))

$$E[f(X)] = \sum (f(k) \cdot P(X=k))$$

(note that the
probabilities are
still weighted
using **X** (not **f(X)**)

*on average, what value can we “expect” **f(X)** to take?*



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LoE

Linearity of **E**xpectation is a powerful property of random variables!



02 - Linearity of Expectation



Random Variables

allow us to represent a quantitative property of a random experiment

EXPECTATION - weighted average of possible outcomes

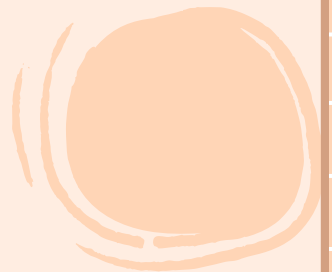
you could use “brute force” and use the formula for expectation ($E[X] = \sum (x * P(x))$)

sometimes, just applying the formula can be messy, so LoE comes in handy

LINEARITY OF EXPECTATION (LoE) *is one important property*

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is
the sum of their expected values





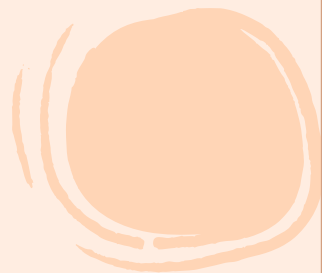
02 - Linearity of Expectation



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is
the sum of their expected values

*this gives us a helpful **tool to calculate expectations of complex RVs***





02 - Linearity of Expectation



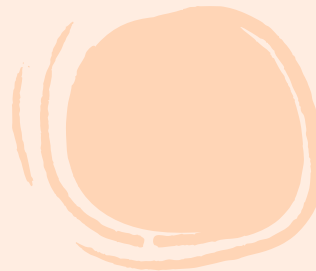
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DECOMPOSE into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$





02 - Linearity of Expectation



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DECOMPOSE into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$

APPLY linearity of expectation

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$



02 - Linearity of Expectation



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

this gives us a helpful tool to calculate expectations of complex RVs

DECOMPOSE into a sum of random variables $X = X_1 + X_2 + \dots + X_n$

APPLY linearity of expectation $E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$

CONQUER and calculate each value $E[X_1] = \dots, E[X_2] = \dots, \dots$



02 - Linearity of Expectation



$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

sometimes, these X_i variables we “decompose” X into are **indicator** random variables

this gives us a helpful tool to calculate expectations of complex RVs

DECOMPOSE into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$

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$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

CONQUER and calculate each value

$$E[X_1] = \dots, E[X_2] = \dots, \dots$$



02 - Linearity of Expectation



***X and Y DON'T have to be independent!

$$E(X+Y) = E(X) + E(Y)$$

the expected value of the sum of 2 random variables is the sum of their expected values

sometimes, these X_i variables we "decompose" X into are **indicator** random variables

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DECOMPOSE into a sum of random variables

$$X = X_1 + X_2 + \dots + X_n$$

APPLY linearity of expectation

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n]$$

CONQUER and calculate each value

$$E[X_1] = \dots, E[X_2] = \dots, \dots$$



02 - Linearity of Expectation



Indicator Random Variables

we can define a *indicator random variable* X for an event A

$$X = \begin{cases} 1 & \text{if event } A \text{ happens} \\ 0 & \text{if event } A \text{ doesn't happen} \end{cases}$$

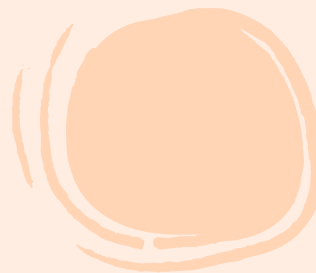
X tells us whether event A will happen \rightarrow so, $P(X = 1) = P(A)$

Note that $E[X] = 1 * P(X=1) + 0 * P(X=0) = P(X=1)$

*this is why indicator RVs
can be really useful when
applying linearity of
expectation!*



**Additional slides for content that
will be covered later in the week!**





02 - Linearity of Expectation



linearity of expectation is special!

$$E[X+Y] = E[X] + E[Y] \text{ but } E[X^2] \neq (E[X])^2$$

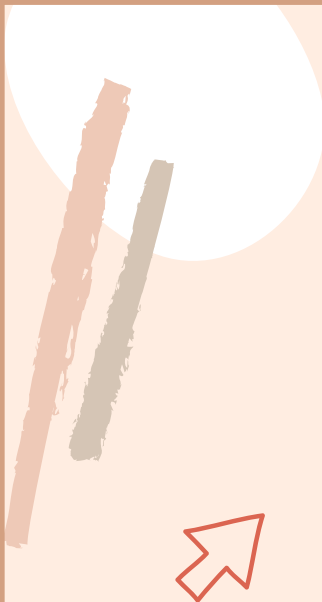
instead...

$$E[g(X)] = \sum (g(x) * P(X=x))$$



Variance

Variance is another property of RVs (like expectation) that measures how much the values in the RV “vary”





03 - Variance



Random Variables

allow us to represent a quantitative property of a random experiment

VARIANCE - how “different” are values from the expectation “on average”

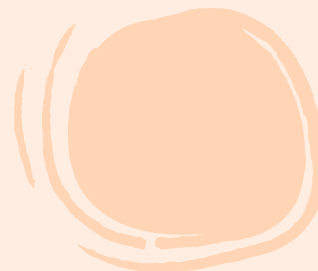
every random variable has some variance

$$\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) * (x - E(X))^2)$$

expected value of the squared distance between each RV outcome and the expected value of RV

add up all the squared distances weighted by their probabilities

$$\text{variance} = (\text{standard deviation})^2$$





03 - Variance



Random Variables

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every random variable has some variance

$$\text{Var}(X) = E[(X - E(X))^2] = \sum_x (P(X=x) * (x - E(X))^2)$$

Properties

$$\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$





03 - Variance



Random Variables

allow us to represent a quantitative property of a random experiment

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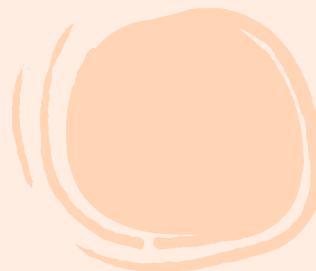
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$$\text{Var}(a \cdot X + b) = a^2 \cdot \text{Var}(X)$$

$$\text{Var}(X) = E[X^2] - (E[X])^2$$





INDEPENDENT RV

What does independence mean for
random variables?





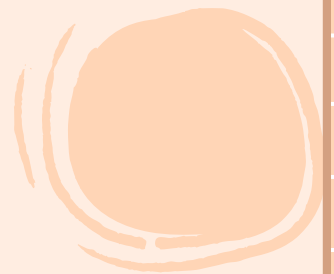
Random Variable Independence



Random variables X and Y are **independent** if, *for all x, y in the ranges of X and Y :*

$$\mathbf{P(X=x, Y=y) = P(X=x) \cdot P(Y=y)}$$

Knowing the value of X doesn't help "guess" what Y is





Random Variable Independence



Random variables X and Y are **independent** if –

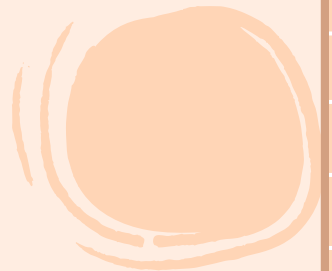
$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of X doesn't help "guess" what Y is

it's a useful property! if X and Y are independent random variables then –

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

$$\text{Var}(X + Y) = \text{Var}[X] + \text{Var}[Y] \quad \textit{Linearity of variance holds}$$





Random Variable Independence



Random variables X and Y are **independent** if –

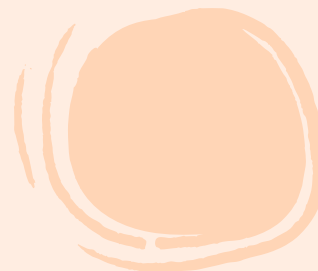
$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

Knowing the value of X doesn't help "guess" what Y is

Additionally, there's **independent and identically distributed** (aka, "i.i.d.") random variables

In addition to independence, i.i.d. random variables also **have the same pmf.**

For example, rolling a die twice, where X is the first roll number and Y is the second roll number





Problems!