CSE 312: Foundations of Computing II Winter 2024

Problem Set 2

Due: Wednesday, January 17, by 11:59pm

Instructions

Solutions format. Every step in your solution should be explained carefully. The logical reasoning behind your solution should be sound and evident from your write-up.

For example, if you are asked to compute the number of ways to permute the set $\{1, 2, 3, 4\}$ that start with 1 or 2, it is not enough to provide the answer 12. A complete approach would explain that (1) we can count separately the permutations starting with 1 and those starting with 2, and that (2) the two sets are disjoint, and hence the overall number is the sum of the numbers of permutations of each type. Then, (3) explain that there are $3!$ permutations of each type. Finally, (4) say that the overall number totals to $2 \cdot 3! = 12$.

A higher number of mathematical symbols in your solution will not make your solution more precise or "better" – what *is* important is that the logical flow is complete and can be followed by the graders. Relying exclusively on mathematical symbols in fact often make the solution less readable. Avoid expressions such as "it easy to see" and "clearly" – just explain these steps.

Also, you may find the following short note (by Francis E. Su at Harvey Mudd) helpful.

Unless a problem states otherwise, you can leave your answer in terms of factorials, combinations, etc., for instance 26^7 or $26!/7!$ or $26 \cdot \binom{26}{7}$ are all good forms for final answers.

Collaboration policy. You are required to submit your own solutions for this problem set. You are allowed to discuss the homework with other students. However, the write up must clearly be your own, and moreover, you must be able to explain your solution at any time. We reserve ourselves the right to ask you to explain your work at any time in the course of this class.

Late policy. You have a total of **six** late days during the quarter, but can only use up to two late days on any one problem set. Please plan ahead, as we will not be willing to add any additional late days except in absolute, verifiable emergencies. The final problem set will not be accepted late (however, it will be due only on Friday of the last week of class).

Solutions submission. You must submit your solution via Gradescope. In particular:

- Submit a *single* PDF containing the solution to all of Tasks 1-6 to Gradescope under "**PSet 2 [Written]**".
- Each numbered task should be solved on its own page (or pages). Follow the prompt on Gradescope to link tasks to your pages.
- Do not write your name on the individual pages – Gradescope will handle that.

- We encourage you to typeset your solution. The homepage provides links to resources to help you doing so using LaTeX. If you do use another tool (e.g., Microsoft Word), we request that you use a proper equation editor to display math (MS Word has one). For example, you should be able to write $\sum_{i=1}^{n} x^i$ instead of x^1 + x^2 + ... + x^n. You can also provide a handwritten solution, as long as it is on a single PDF file that satisfies the above submission format requirements. It is your responsibility to make sure handwritten solutions are readable – we will *not* grade unreadable write-ups.

Academic Integrity: See discussion at the top of Problem Set 1 or in the syllabus.

Task 1 – Circular table [10 pts]

At a committee meeting, all of the 100 people present are to be seated at a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down in their correct place. Use the pigeon-hole principle to show that it is possible to rotate the table so that at least two people are sitting in the correct place. Be sure to specify precisely what the pigeons are, precisely what the pigeonholes are, and precisely what the mapping of pigeons to pigeonholes is.

Task 2 – Thinking Combinatorially [15 pts]

We saw in Lecture 3 that combinatorial proofs can be more elegant than algebraic proofs and also provide insights into an equation that goes beyond algebra. In this task, our goal is to develop the skill and intuition for such proofs. To this end, prove each of the following identities using a *combinatorial argument*; an algebraic solution will be marked substantially incorrect. (Note that $\binom{a}{b}$ is 0 if $b > a$.)

a) (8 points)

$$\binom{a+b}{a} = \sum_{i=0}^{a} \binom{a}{i}\binom{b}{i} .$$

Hint: Start with the left hand side and imagine you are choosing a team of a people from a group of people consisting of a right-handed people and b left-handed people.

b) (7 points)

$$\binom{a}{b} 2^{a-b} = \sum_{i=0}^{a} \binom{a}{i}\binom{i}{b} .$$

Hint: Think about choosing a committee from a group of a people, of which b are leaders.

Task 3 – Stuff into stuff [12 pts]

a) (4 points) We have 10 people and 30 rooms. How many different ways are there to assign the (distinguishable) people to the (distinguishable) rooms? (Any number of people can go into any of the 30 rooms.)

b) (4 points) We have 20 identical (indistinguishable) apples. How many different ways are there to place the apples into 30 (distinguishable) boxes? (Any number of apples can go into any of the boxes.)

c) (4 points) We have 30 identical (indistinguishable) apples. How many different ways are there to place the apples into 8 (distinguishable) boxes, if each box is required to have at least two apples in it?

Task 4 – Principle of Inclusion and Exclusion [10 pts]

How many positive integers are there less than 1000 that are relatively prime to 100, i.e., have no common factor with 100?

Task 5 – Sample Spaces and Probabilities [18 pts]

For each of the following scenarios first describe the sample space and indicate how big it is (i.e., what its cardinality is) and then answer the question. 3 points each.

a) You flip a fair coin 50 times. What is the probability of exactly 20 heads?

b) You roll 2 fair 6-sided dice, one red and one blue. What is the probability that the sum of the two values showing is 4?

c) You are given a random 5 card poker hand (selected from a single deck, order doesn't matter). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?

d) 20 labeled balls are placed into 10 labeled bins (with each placement equally likely). What is the probability that bin 1 contains exactly 3 balls?

e) There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?

f) You buy ten cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. What is the probability that you have at least one of each type?

Task 6 – Random Questions [15 pts]

a) (5 points) What is the probability that the digit 1 doesn't appear among n digits where each digit is one of (0-9) and all sequences are equally likely?

b) (5 points) Suppose you randomly permute the numbers $1, 2, \ldots, n$, (where $n > 500$). That is, you select a permutation uniformly at random. What is the probability that the number 3 ends up in the 130-th position in the resulting permutation? (For example, in the permutation $1, 3, 2, 5, 4$ of the numbers $1 \ldots 5$, the number 2 is in the 3rd position in the permutation and the number 4 is in the 5th position.)

c) (5 points) A fair coin is flipped n times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)