CSE 312

Foundations of Computing II

Lecture 13: Poisson wrap-up
Continuous RV

Agenda

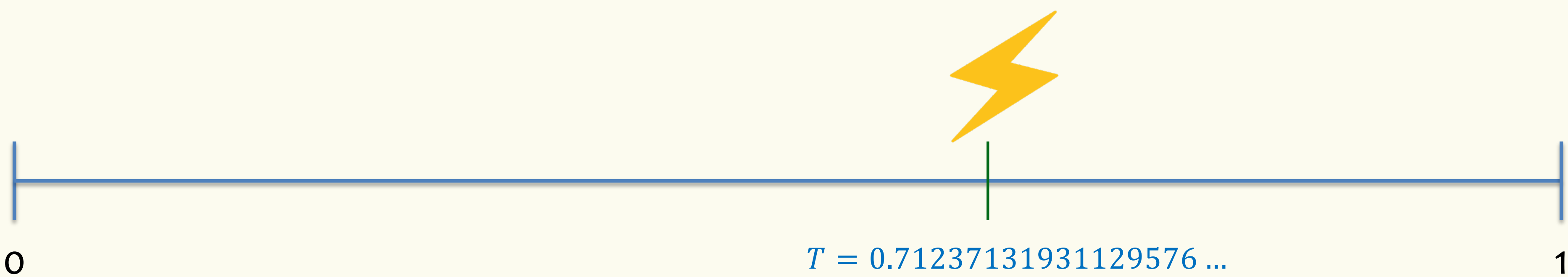
- Wrap-up of Poisson RVs
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function

Often we want to model experiments where the outcome is <u>not</u> discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  – Time measured with infinitesimal precision.

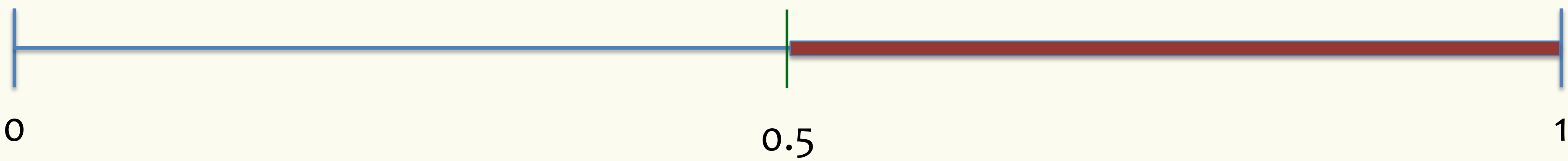


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T \geq 0.5) = ½$$



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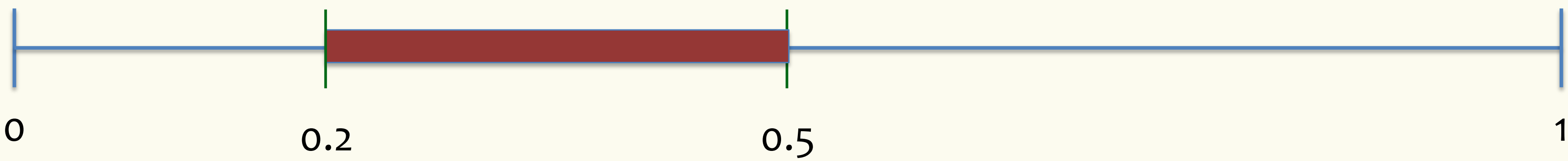
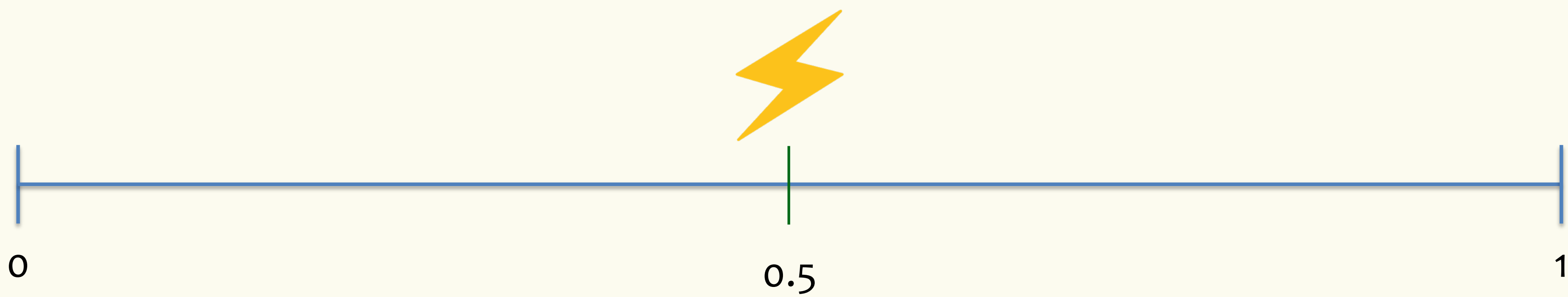


$$P(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3$$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



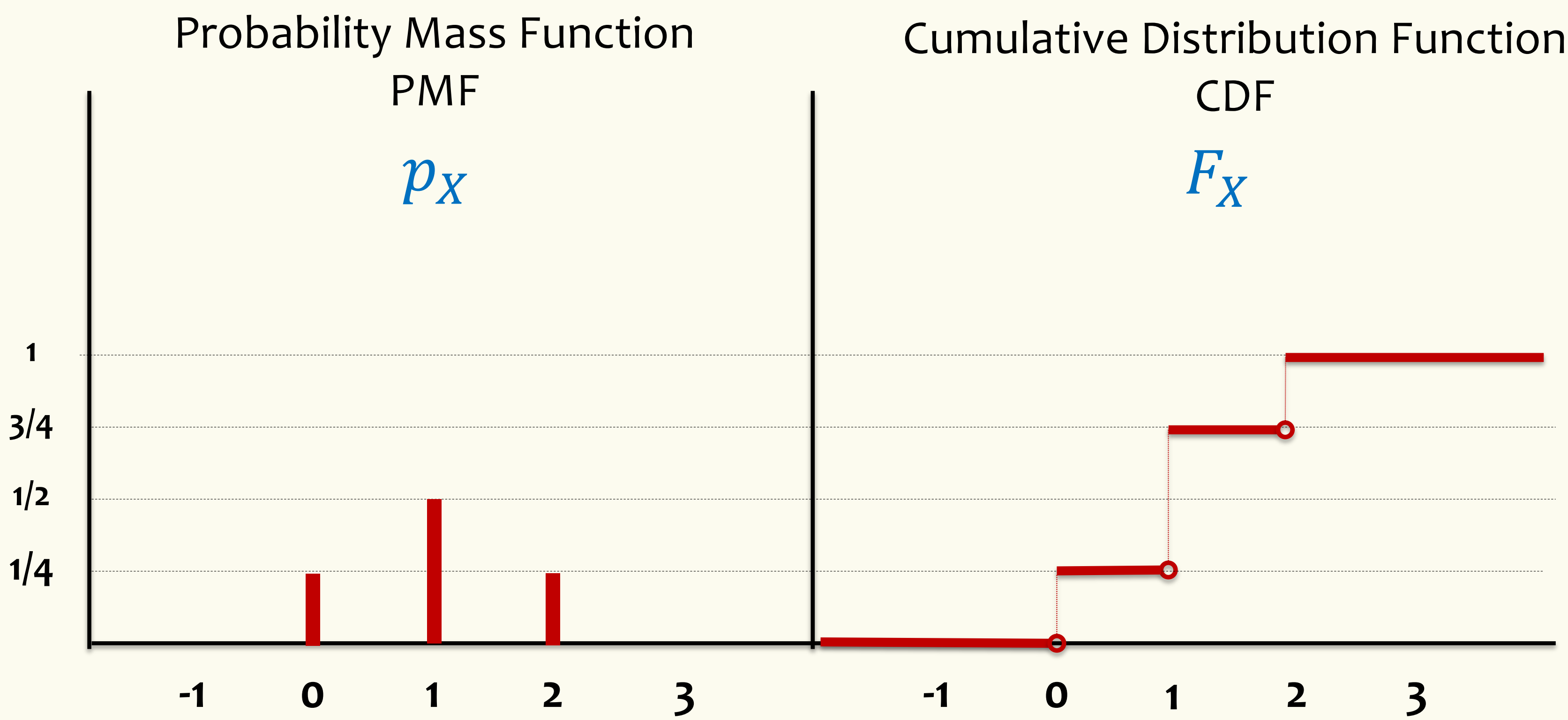
$$P(T = 0.5) = 0$$

Bottom line

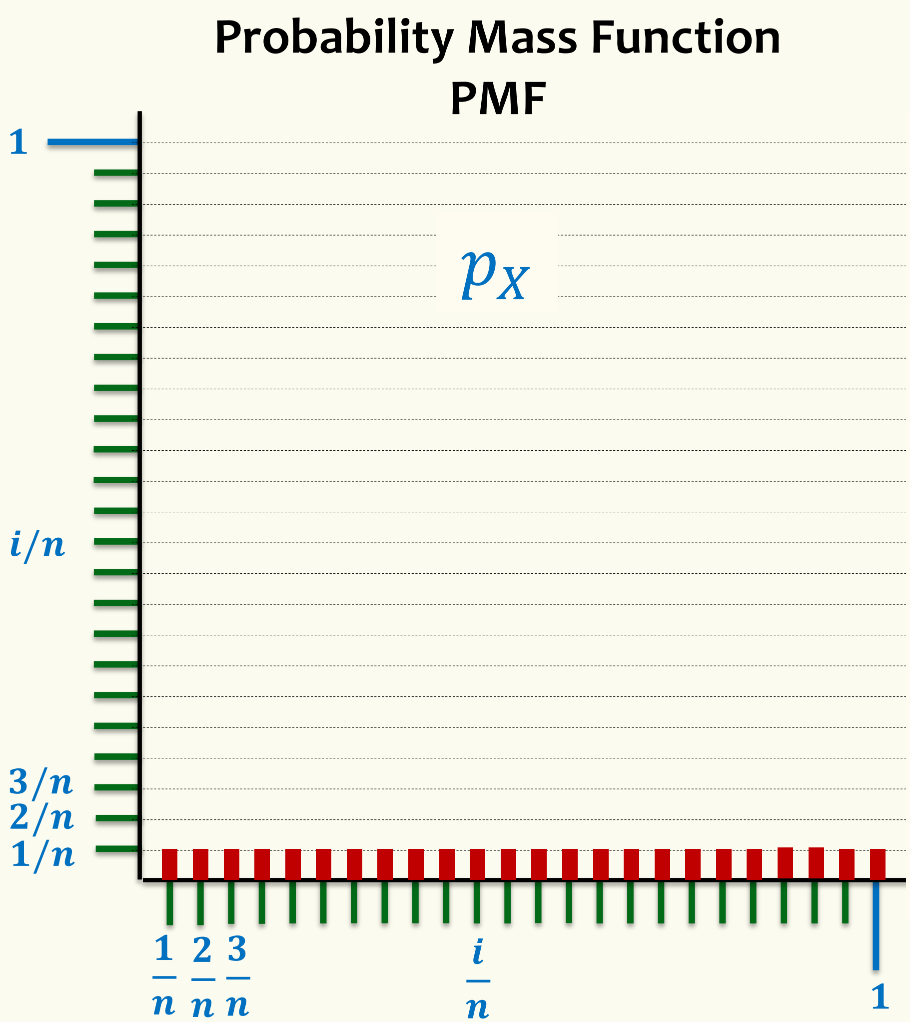
- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $P(T \in [0,1]) = 1$
  - $P(T \in [a, b]) = b - a$
  - ...
- How do we model the behavior of T?

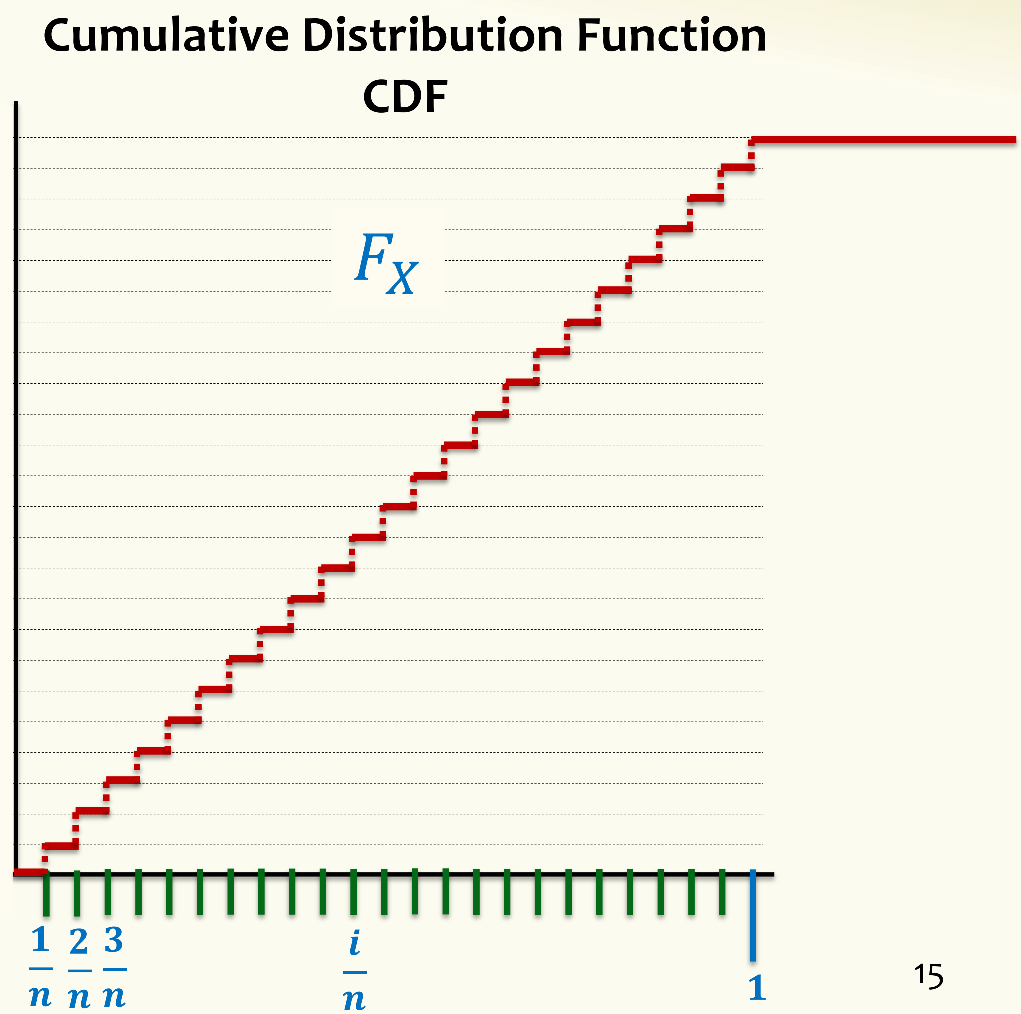
First try: A discrete approximation

Recall: Cumulative Distribution Function (CDF)

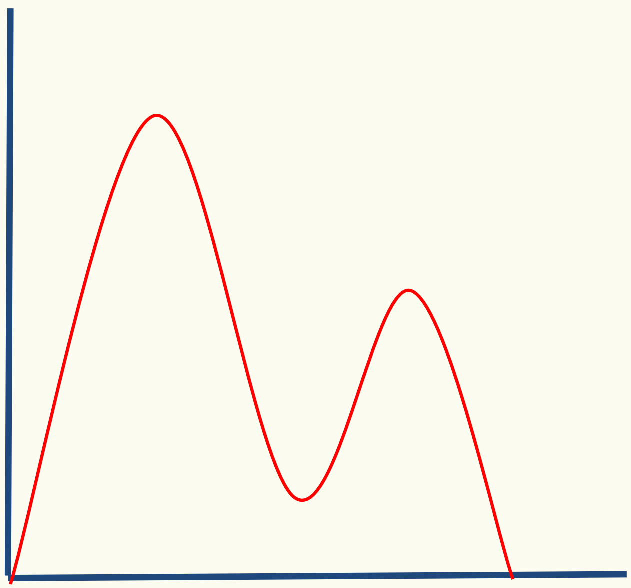


A Discrete Approximation



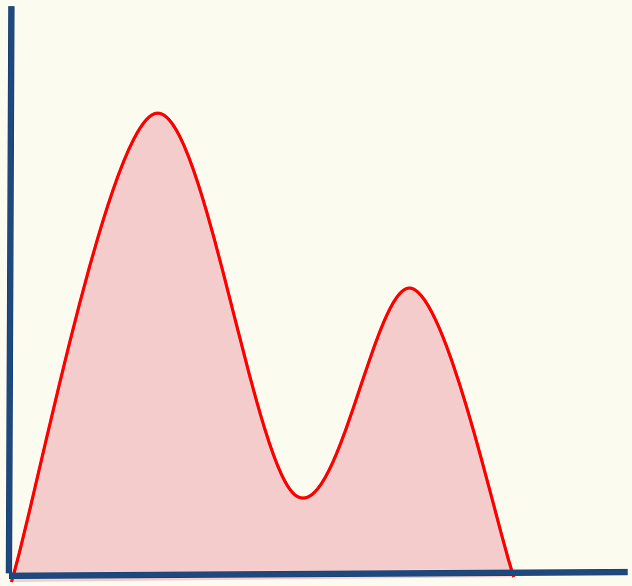


Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X\colon \mathbb{R} \to \mathbb{R}$, such that



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Probability Density Function - Intuition



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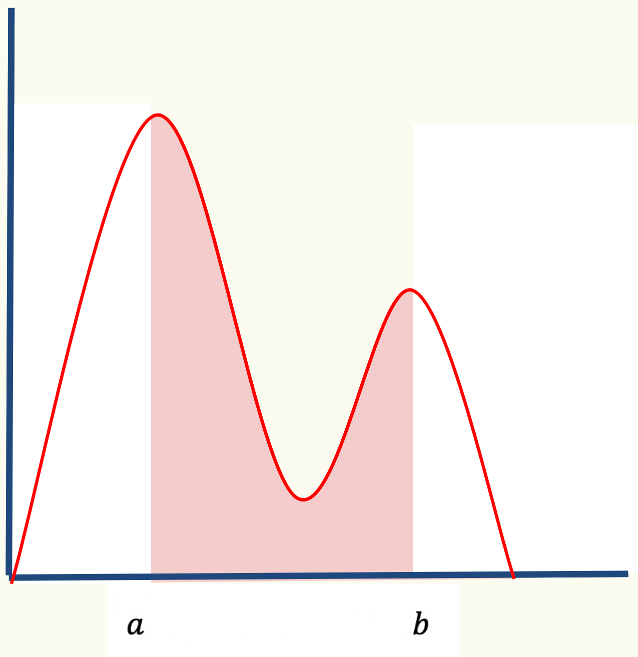

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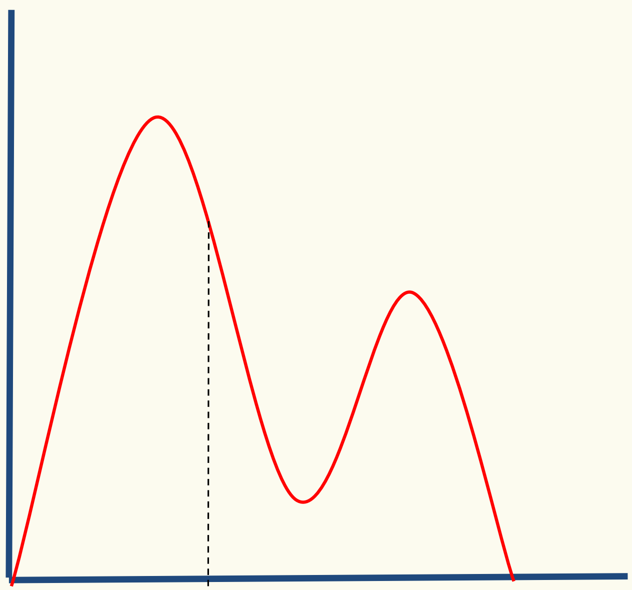

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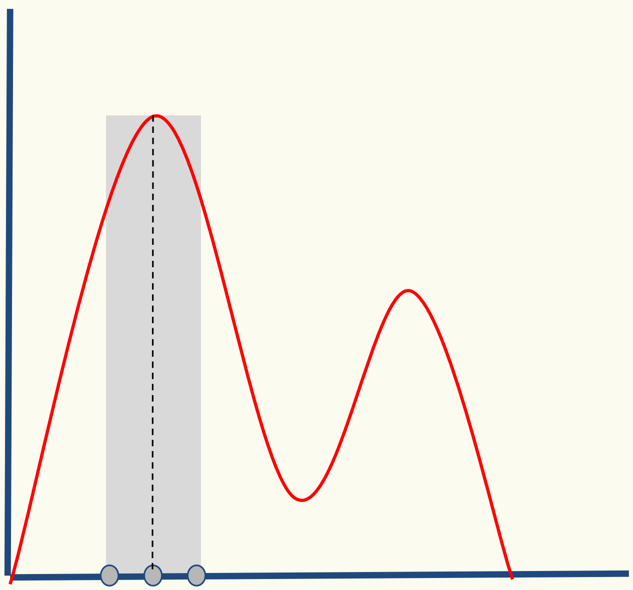
Density ≠ Probability

$f_X(y) \neq 0 \qquad P(X = y) = 0$

Probability Density Function - Intuition



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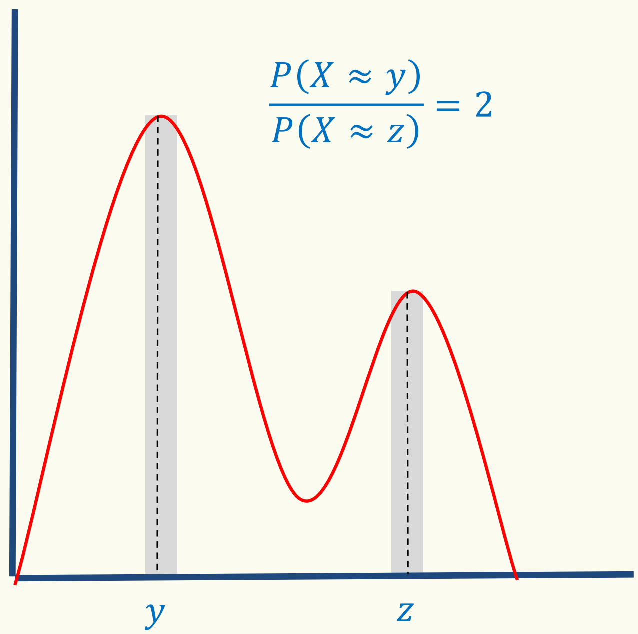
$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

What $f_X(x)$ measures: The local ***rate*** at which probability accumulates

Probability Density Function - Intuition



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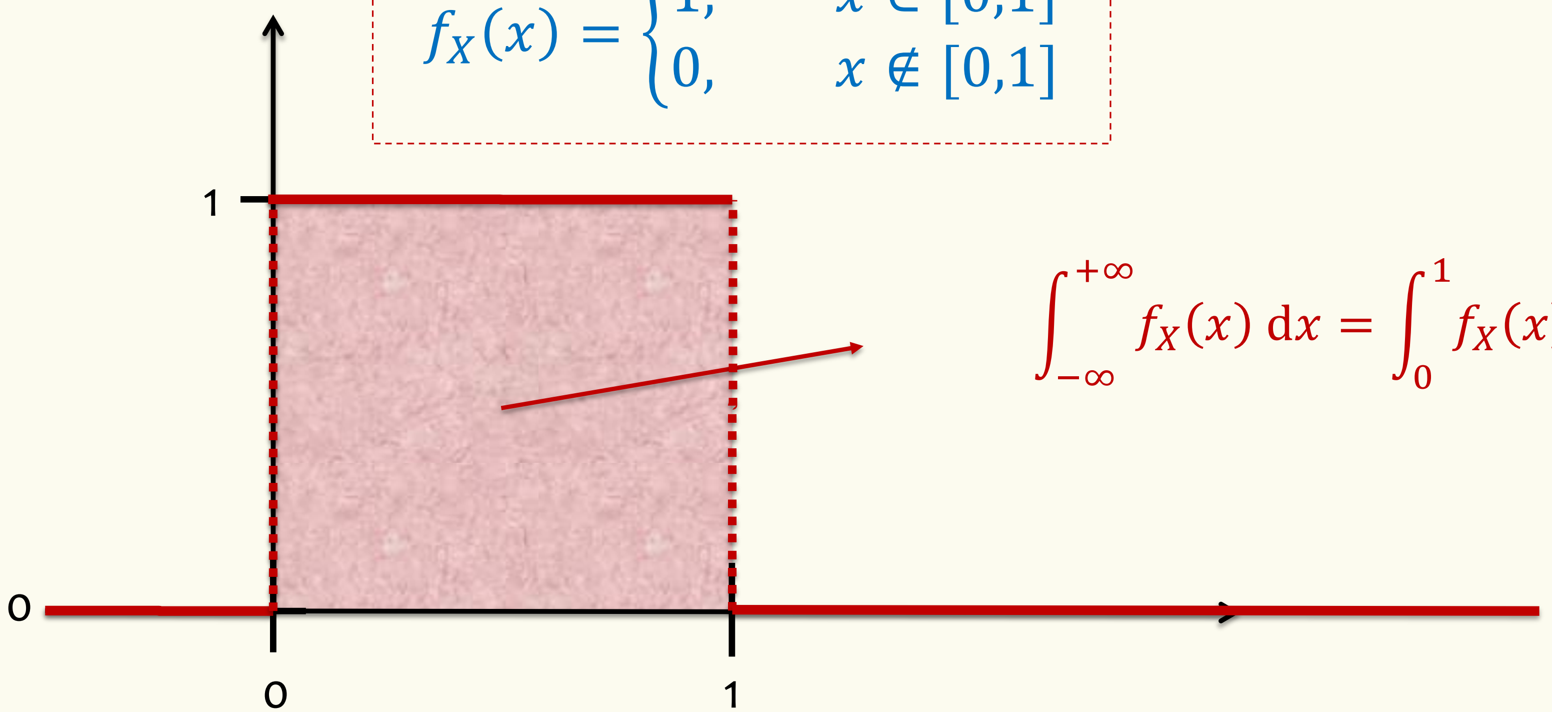
PDF of Uniform RV

$X \sim \mathrm{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event

$X \sim \text{Unif}(0,1)$

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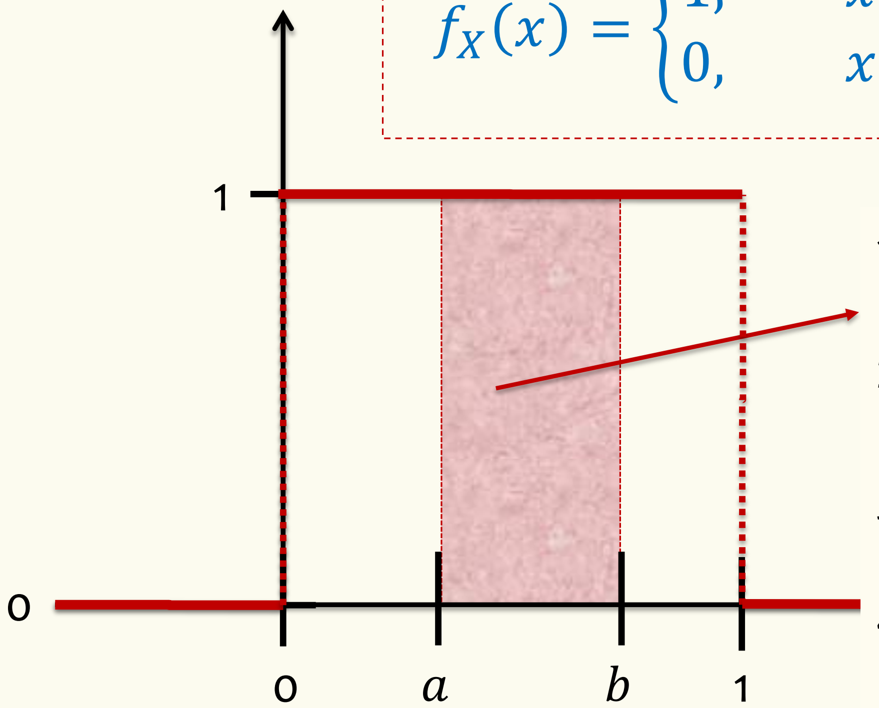

1. If $0 \leq a$ and $a \leq b \leq 1$
   $P(a \leq X \leq b) = b - a$
2. If $a < 0$ and $0 \leq b \leq 1$
   $P(a \leq X \leq b) = b$
3. If $a \geq 0$ and $b > 1$
   $P(a \leq X \leq b) = b - a$
4. If $a < 0$ and $b > 1$
   $P(a \leq X \leq b) = 1$

Poll: **pollev/paulbeame028**

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right

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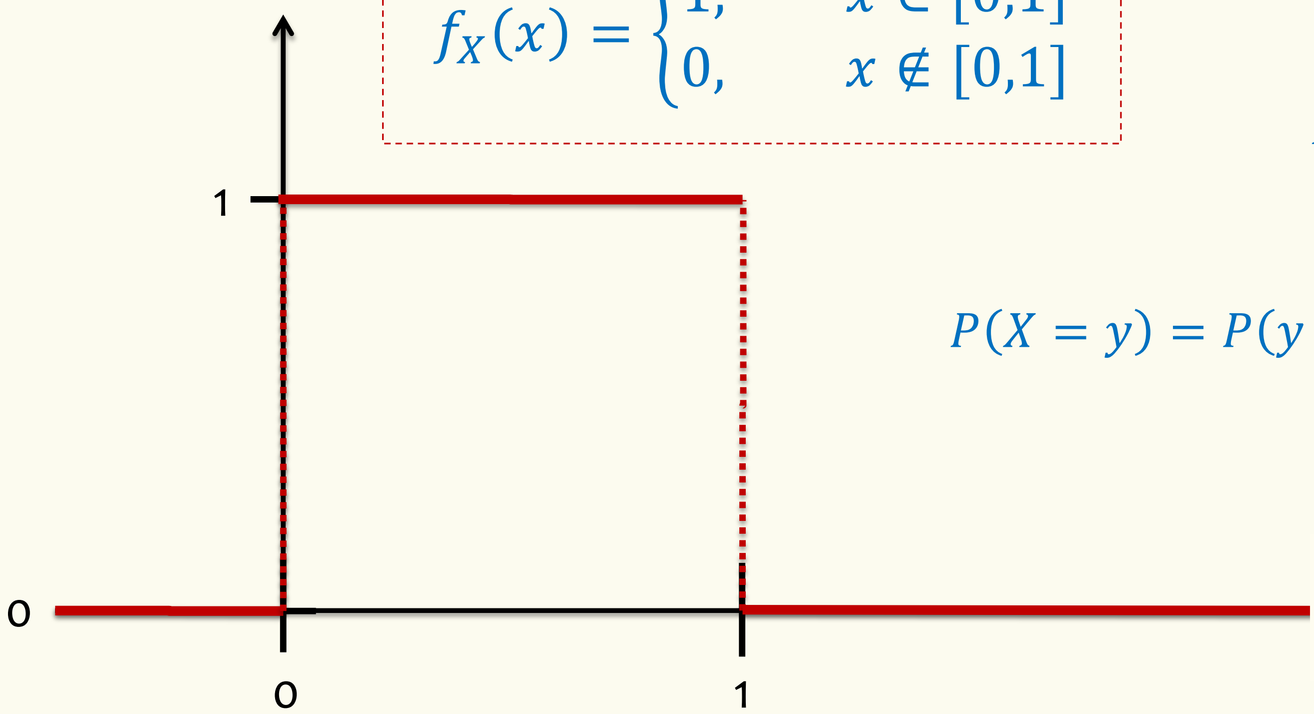

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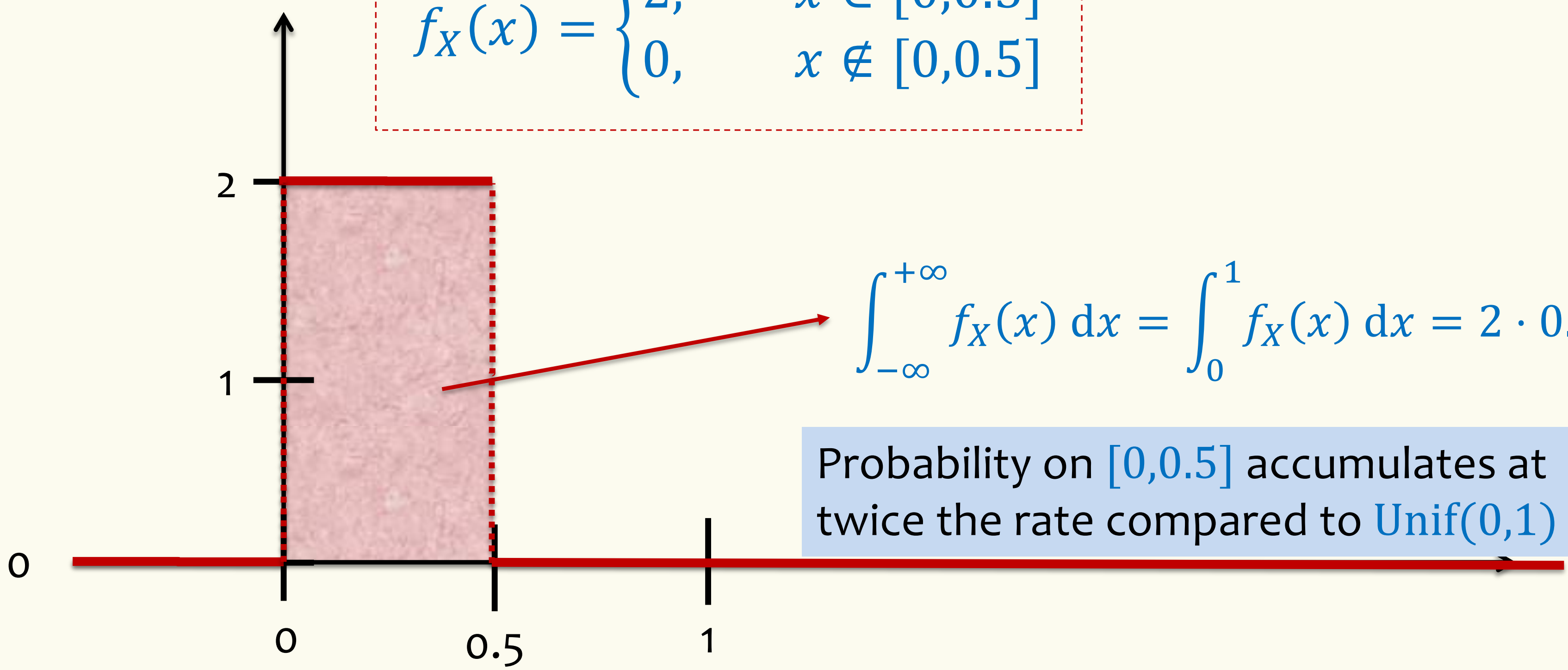
PDF of Uniform RV

Density $\neq$ Probability

$f_X(x) \gg 1$ is possible!

$X \sim \mathrm{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



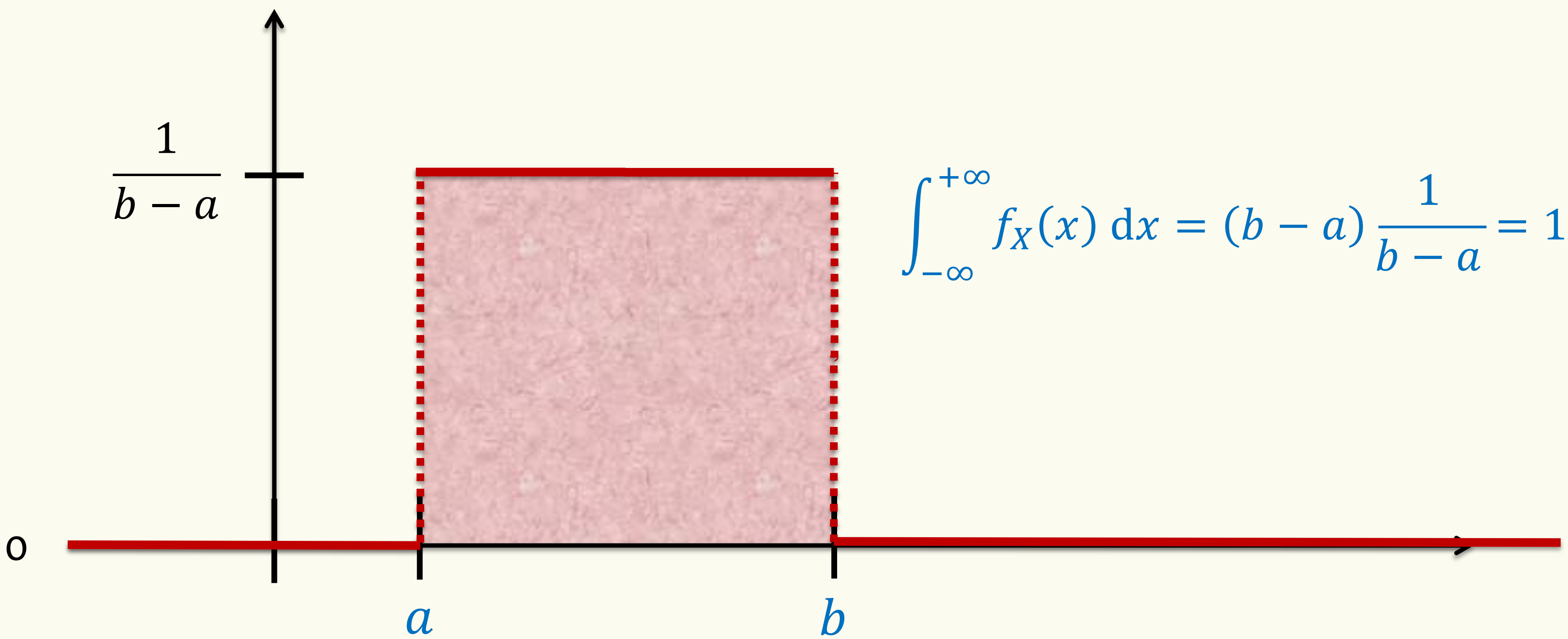
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_{0}^{1} f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Probability on $[0,0.5]$ accumulates at twice the rate compared to $\mathrm{Unif}(0,1)$

Uniform Distribution

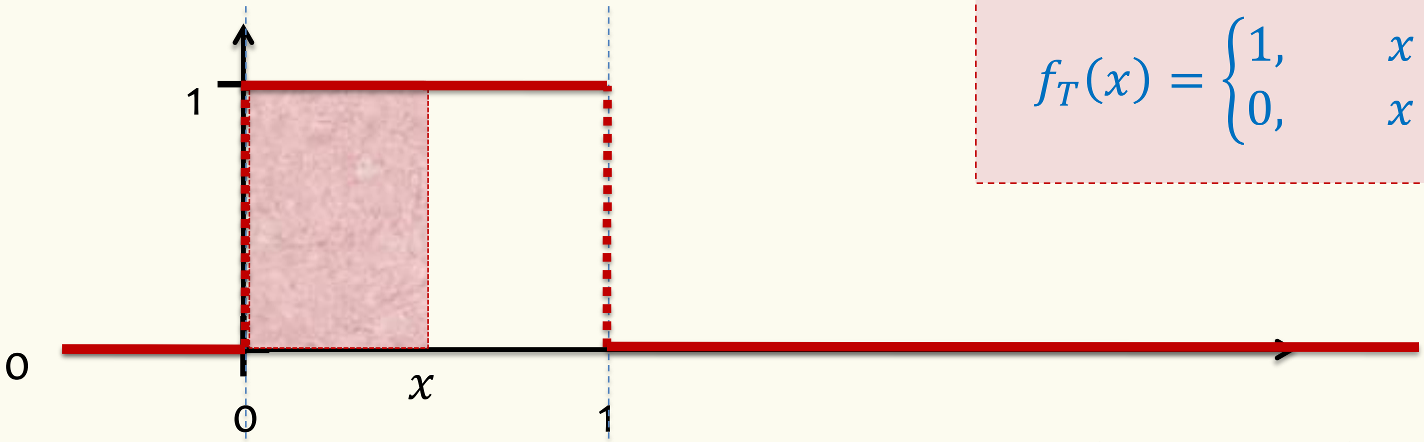
$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

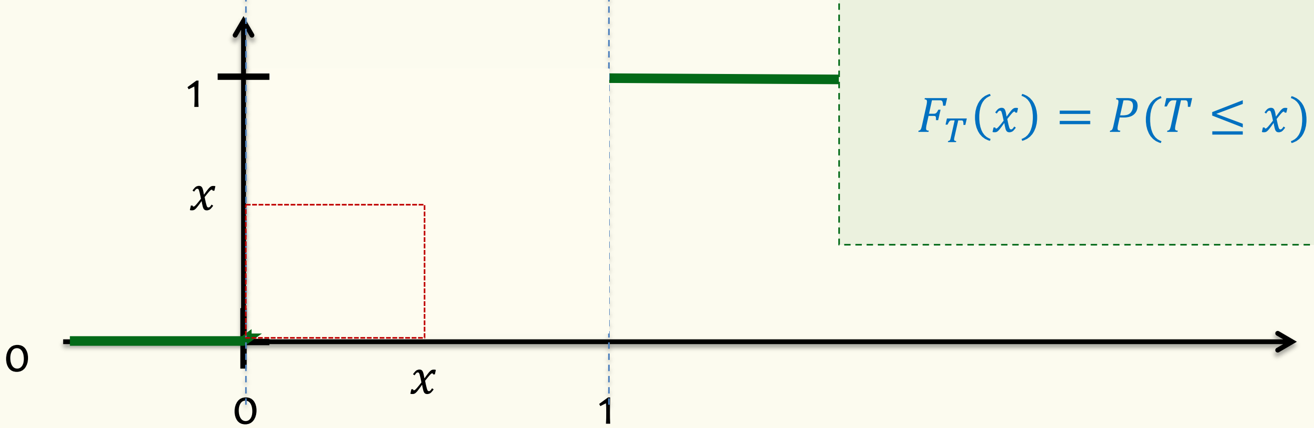
Example. $T \sim \mathrm{Unif}(0,1)$



Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Cumulative Distribution Function



$$F_T(x) = P(T \leq x) = \begin{cases} 0 & x \leq 0 \\ ? & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0 \qquad \lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_{x} p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_{x} g(x)\, p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$