CSE 312

Foundations of Computing II

Lecture 11: Bloom Filters (continued) & Zoo of Random Variables I

Announcements

- Midterm feedback/evaluation is open till next Tuesday. Please take a few mins to fill it out.

Agenda

- Bloom Filters Example & Analysis
- Zoo of Discrete RVs
  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Applications

Bloom Filters

to the rescue

(Named after Burton Howard Bloom)

Bloom Filters

- Stores information about a set of elements $S \subseteq U$.
- Supports two operations:
  1. **`add`**(x) - adds $x \in U$ to the set S
  2. **`contains`**(x) – ideally: true if $x \in S$, false otherwise

Possible *false positives*

Combine with fallback mechanism – can distinguish false positives from true positives with extra cost

Two goals:

1. **Very fast** (ideally constant time) answers to queries "Is $x \in S$?" for any $x \in U$.
2. **Minimal storage** requirements.

Bloom Filters – Ingredients

t_1	1	0	1	0	0
t_2	0	1	0	0	1
t_3	1	0	0	1	0

Basic data structure is a $k \times m$ binary array "the Bloom filter"

- k rows $t_1, \ldots, t_k$, each of size m
- Think of each row as an m-bit vector

k different hash functions $\mathbf{h}_1, \ldots, \mathbf{h}_k : U \to [m]$

We *idealize* each hash function $\mathbf{h}_1$ as assigning each input x to a random output y in [m]

Bloom Filters – Three operations

- Set up Bloom filter for $S = \emptyset$

function INITIALIZE(k, m)
  for $i = 1, \dots, k$: **do**
    t_i = new bit vector of m 0s

- Update Bloom filter for $S \leftarrow S \cup \{x\}$

function ADD(x)
  for $i = 1, \dots, k$: **do**
    $t_i[h_i(x)] = 1$

- Check if $x \in S$

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

Bloom Filters: Example

Bloom filter $\boldsymbol{t}$ of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function INITIALIZE(k, m)
  for $i = 1, \ldots, k$: **do**
    t_i = new bit vector of m 0s

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	0	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\textbf{function } \text{ADD}(x)$$
$$\textbf{for } i = 1, \dots, k\textbf{: do}$$
$$t_i[h_i(x)] = 1$$

add("thisisavirus.com")

h_1("thisisavirus.com") $\rightarrow$ 2

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	0	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\textbf{function } \text{ADD}(x)$$
$$\quad \textbf{for } i = 1, \dots, k\textbf{: do}$$
$$\quad\quad t_i[h_i(x)] = 1$$

add(“thisisavirus.com”)

h_1(“thisisavirus.com”) $\rightarrow$ 2

h_2(“thisisavirus.com”) $\rightarrow$ 1

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	0	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\&\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

add("thisisavirus.com")

h_1("thisisavirus.com") → 2

h_2("thisisavirus.com") → 1

h_3("thisisavirus.com") → 4

Index →	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	0

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{array}{l} \textbf{function } \text{ADD}(x) \\ \quad \textbf{for } i = 1, \dots, k\textbf{: do} \\ \quad\quad t_i[h_i(x)] = 1 \end{array}$$

add(“thisisavirus.com”)

h_1(“thisisavirus.com”) $\rightarrow$ 2

h_2(“thisisavirus.com”) $\rightarrow$ 1

h_3(“thisisavirus.com”) $\rightarrow$ 4

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: Example

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

Since all conditions satisfied, returns True (correctly)

contains("thisisavirus.com")

h_1("thisisavirus.com") $\rightarrow$ 2
h_2("thisisavirus.com") $\rightarrow$ 1
h_3("thisisavirus.com") $\rightarrow$ 4

Index $\rightarrow$	0	1	2	3	4
t_1	0	0	1	0	0
t_2	0	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

$$\begin{aligned}&\textbf{function } \text{ADD}(x)\\&\quad \textbf{for } i = 1, \dots, k\textbf{: do}\\&\quad\quad t_i[h_i(x)] = 1\end{aligned}$$

add("totallynotsuspicious.com")

h_1("totallynotsuspicious.com") → 1

h_2("totallynotsuspicious.com") → 0

h_3("totallynotsuspicious.com") → 4

Index →	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Bloom Filters: False Positives

Bloom filter t of length $\boldsymbol{m}$ = 5 that uses $\boldsymbol{k}$ = 3 hash functions

function CONTAINS(x)
  return $t_1[h_1(x)] == 1 \wedge t_2[h_2(x)] == 1 \wedge \cdots \wedge t_k[h_k(x)] == 1$

Since all conditions satisfied, returns True (incorrectly)

contains("verynormalsite.com")

h_1("verynormalsite.com") $\rightarrow$ 2

h_2("verynormalsite.com") $\rightarrow$ 0

h_3("verynormalsite.com") $\rightarrow$ 4

Index $\rightarrow$	0	1	2	3	4
t_1	0	1	1	0	0
t_2	1	1	0	0	0
t_3	0	0	0	0	1

Analysis: False positive probability

Question: For an element $x \in U$, what is the probability that `contains`(x) returns true if `add`(x) was never executed before?

Probability over what?! Over the choice of the $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_k$

Assumptions for the analysis (somewhat stronger than for ordinary hashing):

- Each $\mathbf{h}_i(x)$ is uniformly distributed in $[m]$ for all x and i
- Hash function outputs for each $\mathbf{h}_i$ are mutually independent (not just in pairs)
- Different hash functions are independent of each other

False positive probability – Events

Assume we perform $\texttt{add}(x_1), \ldots, \texttt{add}(x_n)$
+ $\texttt{contains}(x)$ for $x \notin \{x_1, \ldots, x_n\}$

Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \ldots, \mathbf{h}_i(x_n)\}$

$$P(\text{false positive}) = P(E_1 \cap E_2 \cap \cdots \cap E_k) = \prod_{i=1}^{k} P(E_i)$$

$\mathbf{h}_1, \ldots, \mathbf{h}_k$ independent

False positive probability – Events

Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \dots, \mathbf{h}_i(x_n)\}$

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c) = \sum_{z=1}^{m} \underbrace{P(\mathbf{h}_i(x) = z)}_{\text{LTP}} \cdot P(E_i^c \mid \mathbf{h}_i(x) = z)$$

LTP

False positive probability – Events

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c \mid \mathbf{h}_i(x) = z) = P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z \mid \mathbf{h}_i(x) = z)$$

Independence of values of $\boldsymbol{h}_i$ on different inputs

$$= P(\mathbf{h}_i(x_1) \neq z, \dots, \mathbf{h}_i(x_n) \neq z)$$

$$= \prod_{j=1}^{n} P(\mathbf{h}_i(x_j) \neq z)$$

Outputs of $\boldsymbol{h}_i$ uniformly spread

$$= \prod_{j=1}^{n} \left(1 - \frac{1}{m}\right) = \left(1 - \frac{1}{m}\right)^n$$

$$P(E_i^c) = \sum_{z=1}^{m} P(\mathbf{h}_i(x) = z) \cdot P(E_i^c \mid \mathbf{h}_i(x) = z) = \left(1 - \frac{1}{m}\right)^n$$

False positive probability – Events

Event E_i holds iff $\mathbf{h}_i(x) \in \{\mathbf{h}_i(x_1), \ldots, \mathbf{h}_i(x_n)\}$

Event E_i^c holds iff $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_1)$ and … and $\mathbf{h}_i(x) \neq \mathbf{h}_i(x_n)$

$$P(E_i^c) = \left(1 - \frac{1}{m}\right)^n$$

$$\Rightarrow \quad \mathrm{FPR} = \prod_{i=1}^{k} \left(1 - P(E_i^c)\right) = \left(1 - \left(1 - \frac{1}{m}\right)^n\right)^k$$

False Positivity Rate – Example

$$\text{FPR} = \left(1 - \left(1 - \frac{1}{m}\right)^{n}\right)^{k}$$

e.g., $n = 5{,}000{,}000$

$k = 30$

$m = 2{,}500{,}000$

FPR = 1.28%

Comparison with Hash Tables - Space

- Google storing 5 million URLs, each URL 40 bytes.
- Bloom filter with $k = 30$ and $m = 2{,}500{,}000$

Hash Table

(optimistic)
$5{,}000{,}000 \times 40B = 200\text{MB}$

Bloom Filter

$2{,}500{,}000 \times 30 = 75{,}000{,}000$ bits

< 10 MB

Time

- Say avg user visits 102,000 URLs in a year, of which 2,000 are malicious.
- 0.5 seconds to do lookup in the database, 1ms for lookup in Bloom filter.
- Suppose the false positive rate is 3%

0.5 seconds DB lookup

false positives

$$\underbrace{1\text{ms}}_{\text{Bloom filter lookup}} + \frac{\overbrace{100000\times 0.03}^{\text{false positives}}\times 500\text{ms} + \underbrace{2000}_{\text{malicious URLs}}\times 500\text{ ms}}{\underbrace{102000}_{\text{total URLs}}} \approx 25.51\text{ms}$$

total URLs

malicious URLs

Bloom filter lookup

Bloom Filters typical of….

… randomized algorithms and randomized data structures.

- **Simple**
- **Fast**
- **Efficient**
- **Elegant**
- **Useful!**

Brain Break

Motivation for "Named" Random Variables

Random Variables that show up all over the place.

- Easily solve a problem by recognizing it's a special case of one of these random variables.

Each RV introduced today will show:

- A general situation it models
- Its name and parameters
- Its PMF, Expectation, and Variance
- Example scenarios you can use it

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  - Uniform Random Variables
  - Bernoulli Random Variables
  - Binomial Random Variables
  - Applications

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

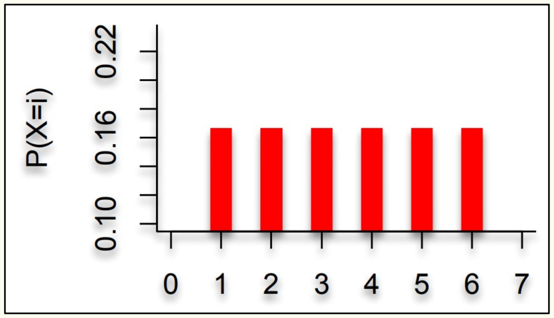
Notation:

PMF:

Expectation:

Variance:

Example: value shown on one roll of a fair die



Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (integer) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \mathrm{Unif}(a, b)$

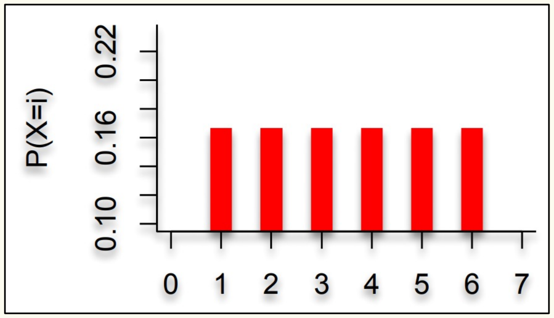
PMF: $\mathrm{P}(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathbb{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1,6)$:

- $P(X = i) = 1/6$
- $\mathbb{E}[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



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Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation:

Variance:

Poll:
pollev.com/rachel312

	Mean	Variance
A.	p	p
B.	p	$1 - p$
C.	p	$p(1 - p)$
D.	p	p^2

Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \text{Ber}(p)$

PMF: $P(X = 1) = p,\ P(X = 0) = 1 - p$

Expectation: $\mathbb{E}[X] = p$ Note: $\mathbb{E}[X^2] = p$

Variance: $\text{Var}(X) = \mathbb{E}[X^2] - \boxed{\mathbb{E}[X]}^2 = p - p^2 = p(1-p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails
- Any indicator RV

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Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \mathrm{Ber}(p)$.

X is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Examples:

- # of heads in n coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table

Poll:
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$P(X = k)$

A. $p^k(1-p)^{n-k}$

B. np

C. $\binom{n}{k} p^k (1-p)^{n-k}$

D. $\binom{n}{n-k} p^k (1-p)^{n-k}$

Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \text{Ber}(p)$.
X is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Notation: $X \sim \text{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation:

Variance:

Poll:
pollev.com/Rachel312

	Mean	Variance
A.	p	p
B.	np	$np(1-p)$
C.	np	np^2
D.	np	n^2p

Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \mathrm{Ber}(p)$.
X is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathbb{E}[X] = np$

Variance: $\mathrm{Var}(X) = np(1-p)$

Mean, Variance of the Binomial

"i.i.d." is a commonly used phrase.
It means "independent & identically distributed"

If $Y_1, Y_2, \dots, Y_n \sim \mathrm{Ber}(p)$ and independent (i.i.d.), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \mathrm{Bin}(n, p)$

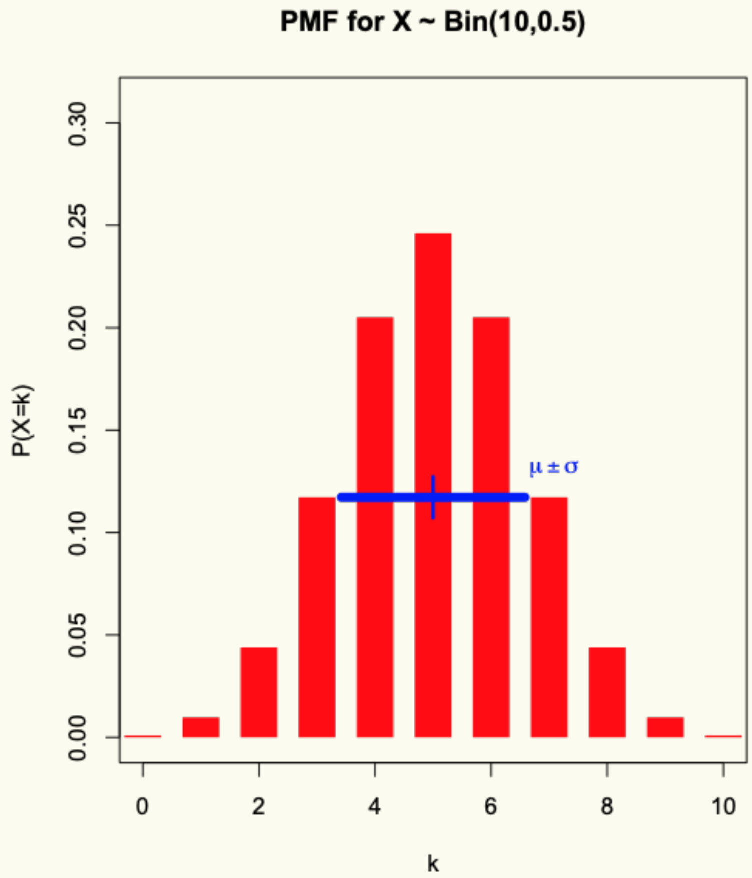
Claim $\mathbb{E}[X] = np$

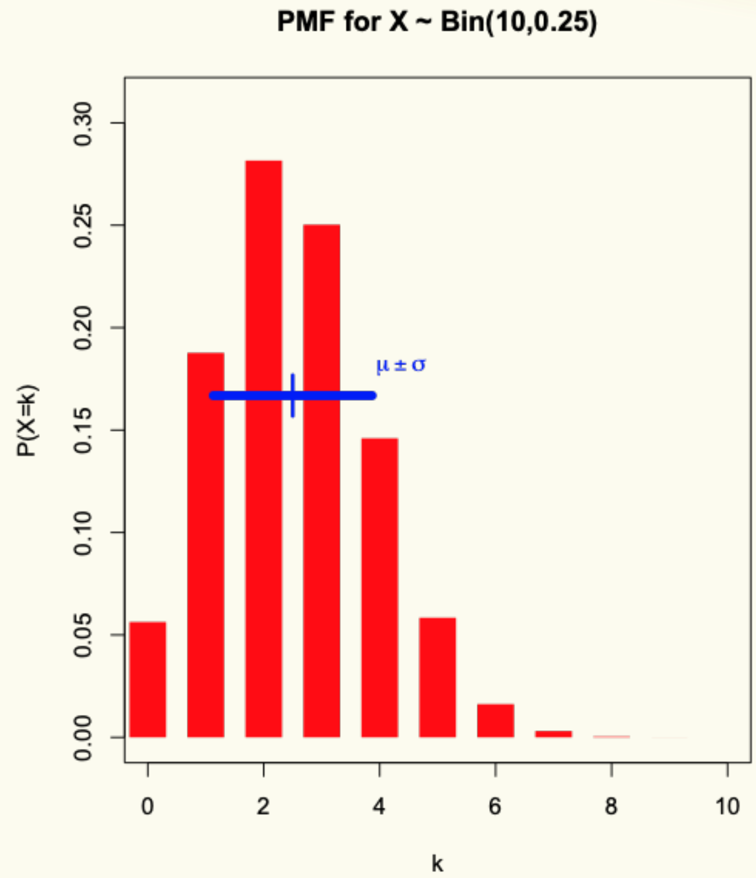
$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} \mathbb{E}[Y_i] = n\mathbb{E}[Y_1] = np$$

Claim $\mathrm{Var}(X) = np(1-p)$

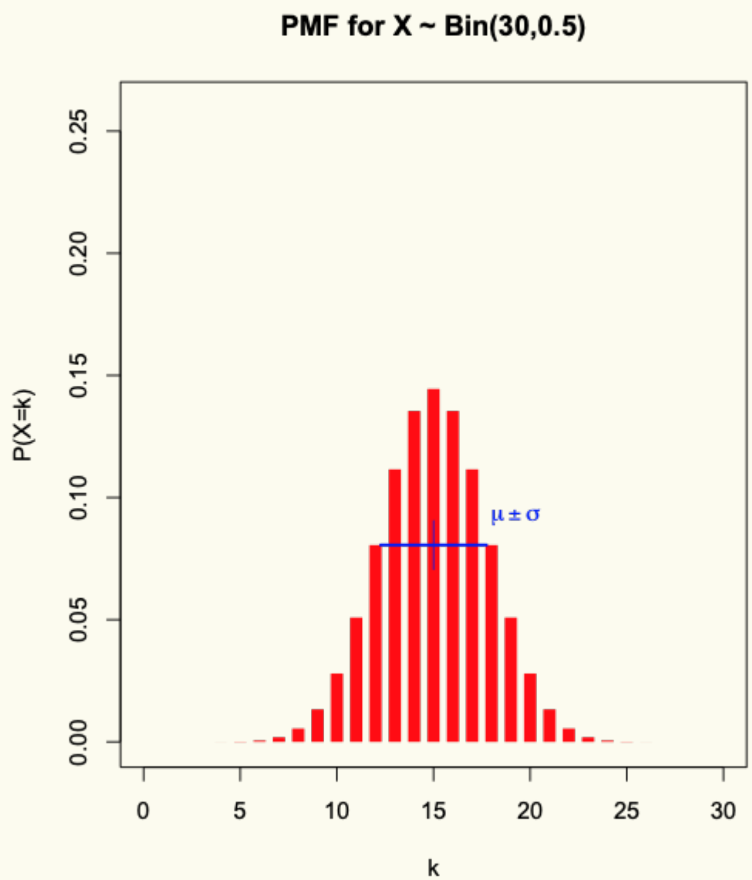
$$\mathrm{Var}(X) = \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathrm{Var}(Y_i) = n\mathrm{Var}(Y_1) = np(1-p)$$

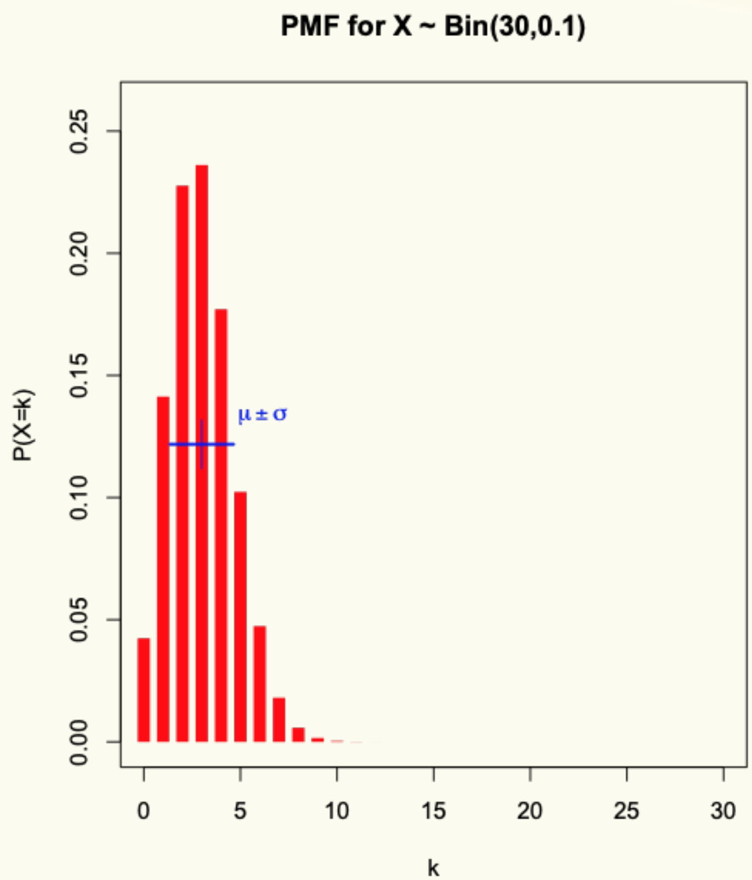
Binomial PMFs





Binomial PMFs





Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits).

Let X be the number of corrupted bits.

What is $\mathbb{E}[X]$?

Poll:
pollev.com/rachel312

a. 1022.99
b. 1.024
c. 1.02298
d. 1
e. Not enough information to compute

Welcome to the Zoo! (today) 🦍🐘🦁🐅🦓🐪🦒

$X \sim \mathrm{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$\mathbb{E}[X] = \frac{a + b}{2}$$

$$\mathrm{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \mathrm{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$\mathbb{E}[X] = p$$

$$\mathrm{Var}(X) = p(1 - p)$$

$X \sim \mathrm{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mathbb{E}[X] = np$$

$$\mathrm{Var}(X) = np(1 - p)$$