CSE 312

Foundations of Computing II

Lecture 7: Random Variables

Review Chain rule & independence

Theorem. (Chain Rule) For events $A_1, A_2, \dots, A_n$,

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

Definition. Two events A and A are (statistically) **independent** if

$$P(A \cap B) = P(A) \cdot P(B).$$

"Equivalently." $P(A|B) = P(A)$.

Definition. Two events A and B are **independent conditioned on** C if $P(C) \neq 0$ and $P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C)$.

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

Random Variables

Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its *range/support*

Two common notations: $X(\Omega)$ or Ω_X

Example. Two coin flips: $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

$X =$ number of heads in two coin flips

$X(\mathrm{HH}) = 2 \quad X(\mathrm{HT}) = 1 \quad X(\mathrm{TH}) = 1 \quad X(\mathrm{TT}) = 0$

range (or support) of X is $X(\Omega) = \{0,1,2\}$

Another RV Example

20 different balls labeled 1, 2, …, 20 in a jar

- Draw a subset of 3 from the jar uniformly at random
- Let X = maximum of the 3 numbers on the balls
  - Example: $X(\{2, 7, 5\}) = 7$
  - Example: $X(\{15, 3, 8\}) = 15$

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How large is $|X(\Omega)|$?

A. 20^3
B. 20
C. 18
D. $\binom{20}{3}$

Random Variables

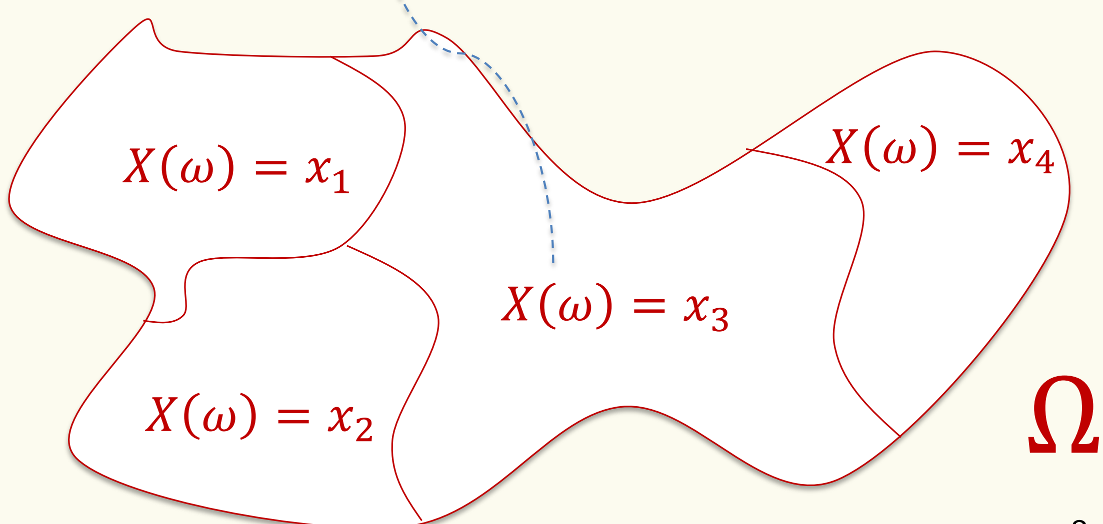
Definition. For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $P(X = x) = P(\{X = x\})$

Random variables **partition** the sample space.

$\Sigma_{x \in X(\Omega)} P(X = x) = 1$



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Example. Two coin flips: $\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

$X =$ number of heads in two coin flips $\quad \Omega_X = X(\Omega) = \{0,1,2\}$

$$P(X = 0) = \frac{1}{4} \qquad P(X = 1) = \frac{1}{2} \qquad P(X = 2) = \frac{1}{4}$$

The RV X yields a new probability distribution with sample space $\Omega_X \subset \mathbb{R}$!

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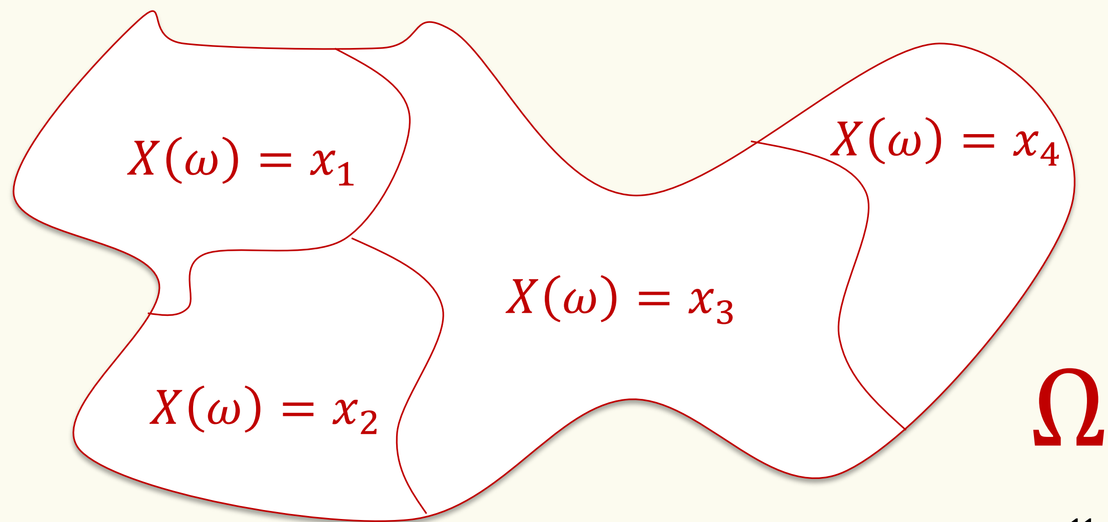
Probability Mass Function (PMF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the function $p_X: \Omega_X \to \mathbb{R}$ defined by $p_X(x) = P(X = x)$ is called the **probability mass function (PMF)** of X

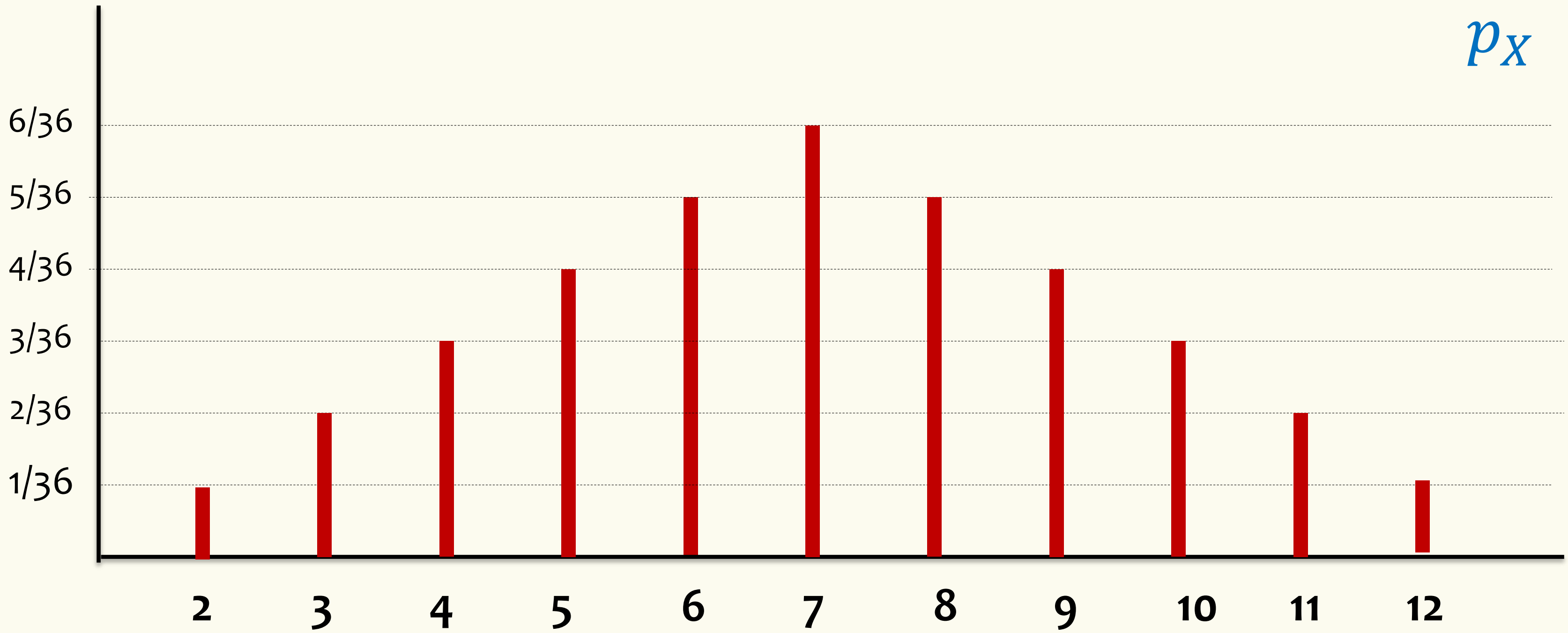
Random variables **partition** the sample space.

$$\left(\sum_{x \in \Omega_X} \underbrace{P(X = x)}_{p_X(x)} \right) = 1$$



Example – Two Fair Dice

X = sum of two dice throws



Example – Number of Heads

We flip n coins, independently, each heads with probability p

$\Omega = \{\mathrm{HH} \cdots \mathrm{HH}, \mathrm{HH} \cdots \mathrm{HT}, \mathrm{HH} \cdots \mathrm{TH}, \ldots, \mathrm{TT} \cdots \mathrm{TT}\}$

$X =$ # of heads

$$p_X(k) = P(X = k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

# of sequences with k heads

Prob of sequence w/ k heads


gifak-net
h_2
h_1
b
a
S=a*h1
S=b*h2

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Events concerning RVs

We already defined $P(X = x) = P(\{X = x\})$ where

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

Sometimes we want to understand other events involving RV X

– e.g. $\{X \leq x\} = \{\omega \in \Omega \mid X(\omega) \leq x\}$ which makes sense for any $x \in \mathbb{R}$

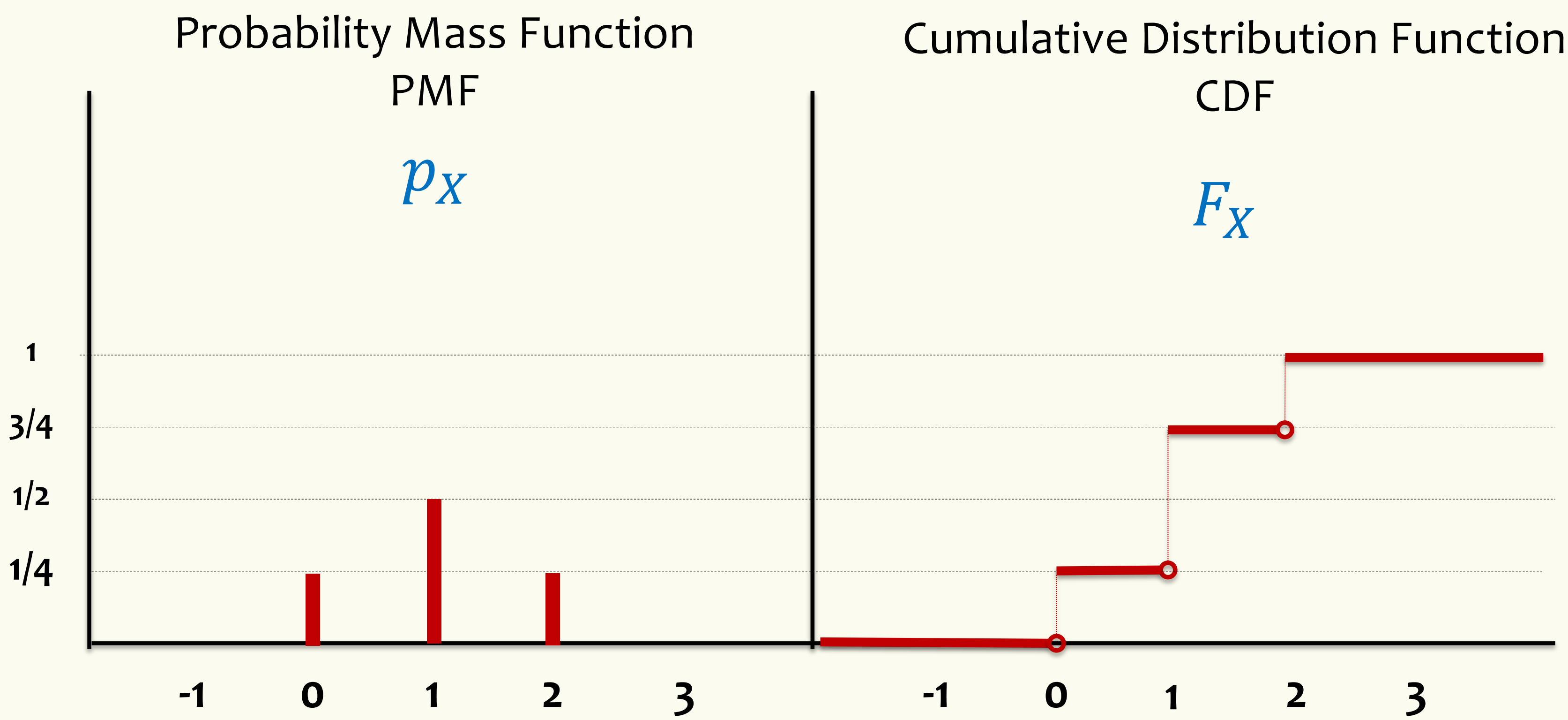
Cumulative Distribution Function (CDF)

Definition. For a RV $X : \Omega \to \mathbb{R}$, the cumulative distribution function of X is the function $F_X : \mathbb{R} \to [0,1]$ that specifies for any real number x, the probability that $X \leq x$.

That is, F_X is defined by $F_X(x) = P(X \leq x)$

Example – Two fair coin flips

X = number of heads



Events concerning RVs

We already defined $P(X = x) = P(\{X = x\})$ where

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More generally…

– We could take any predicate $Q(\cdot)$ defined on the real numbers, and consider an event $\{Q(X)\} = \{\omega \in \Omega \mid Q\big(X(\omega)\big) \text{ is true}\}$

– If $Q(\cdot,\cdot)$ is a predicate of two real numbers and X and Y are RVs both defined on Ω then $\{Q(X, Y)\} = \{\omega \in \Omega \mid Q\big(X(\omega), Y(\omega)\big) \text{ is true}\}$

– The same thing works for properties of even more RVs

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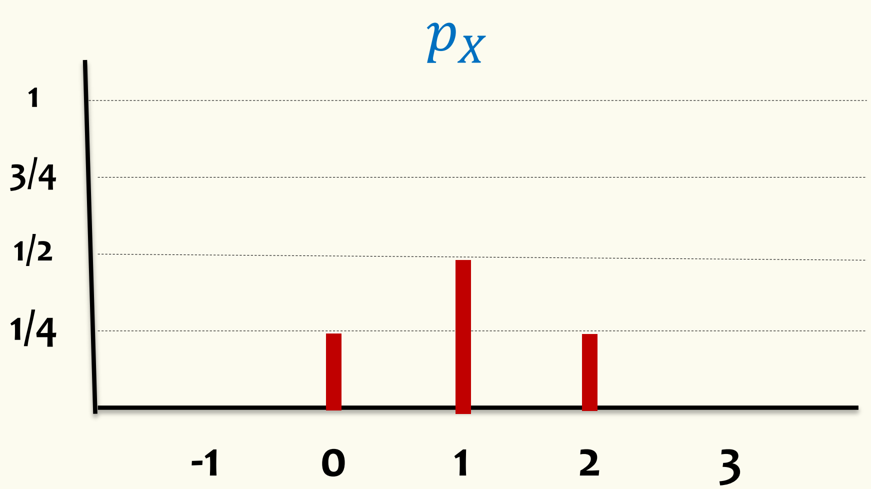
Expectation (Idea)

Example. Two fair coin flips

$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

$X =$ number of heads



- If we chose samples from Ω over and over repeatedly, how many heads would we expect to see per sample from Ω?
  - The idealized number, not the average of actual numbers seen (which will vary from the ideal)

Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Expected Value

Definition. The expected value of a (discrete) RV X is

$$\mathbb{E}[X] = \sum_x x \cdot p_X(x) = \sum_x x \cdot P(X = x)$$

Example. Value X of rolling one fair die

$$p_X(1) = p_X(2) = \cdots = p_X(6) = \frac{1}{6}$$

$$\mathbb{E}[X] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5$$

For the equally-likely outcomes case, this is just the average of the possible outcomes!

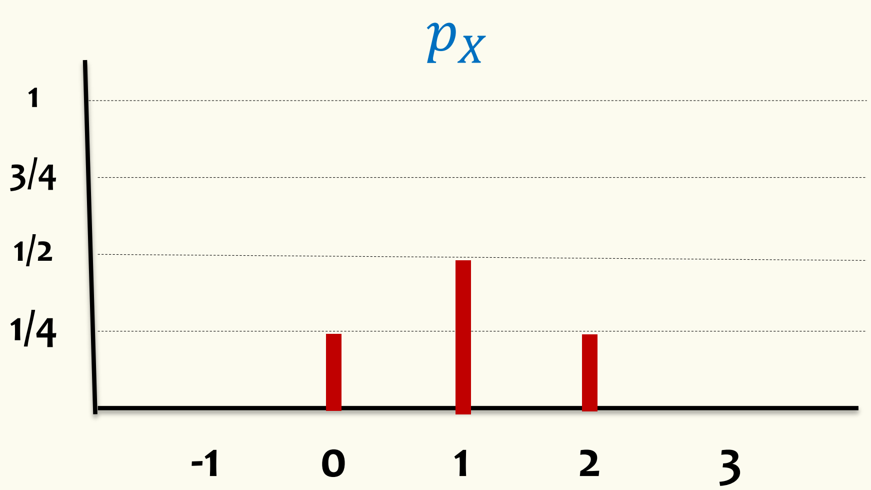
Expectation

Example. Two fair coin flips

$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

What is $\mathbb{E}[X]$?

$X =$ number of heads



$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

0

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, on average, per game, when repeatedly playing?"

Answer: $\mathbb{E}[X]$

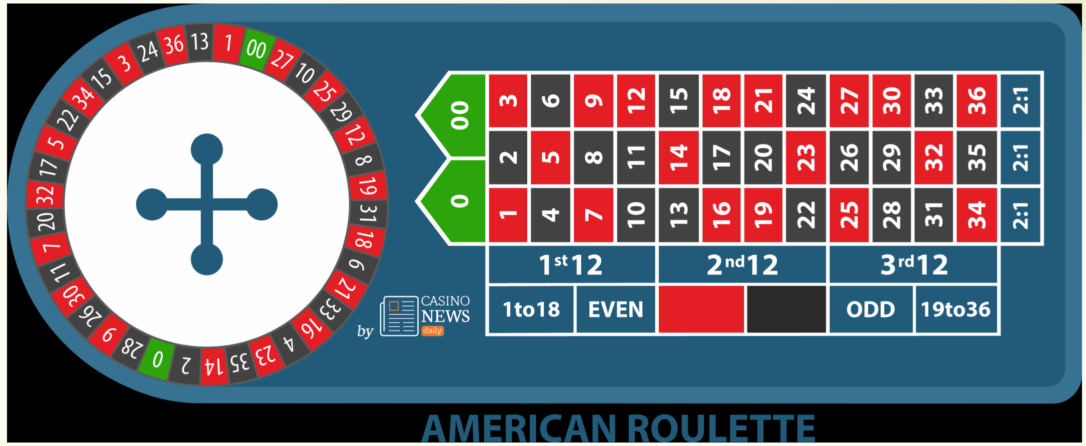
Roulette (USA)

Ω:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00



Note 0 and 00 are not EVEN

RVs for gains from some bets:

RV RED: If Red number turns up $+1$, if Black number, 0, or 00 turns up -1

$$\mathbb{E}[\text{RED}] = (+1)\cdot\frac{18}{38} + (-1)\cdot\frac{20}{38} = -\frac{2}{38} \approx -5.26\%$$

RV $1^{st}12$: If number 1-12 turns up $+2$, if number 13-36, 0, or 00 turns up -1

$$\mathbb{E}[1^{st}12] = (+2)\cdot\frac{12}{38} + (-1)\cdot\frac{26}{38} = -\frac{2}{38} \approx -5.26\%$$

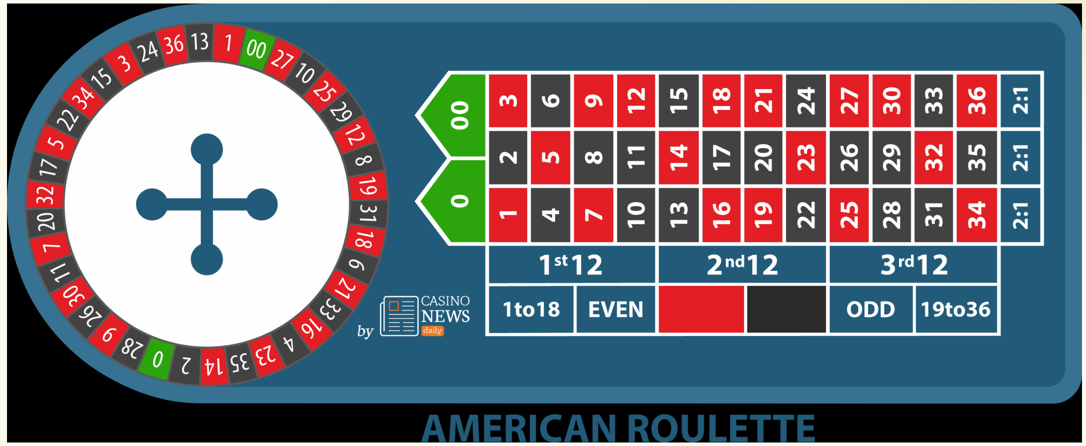
Roulette (USA)

Ω:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00



Note 0 and 00 are not EVEN

An even worse bet:

RV BASKET: If 0, 00, 1, 2, or 3 turns up +6 otherwise -1

$$\mathbb{E}[\text{BASKET}] = (+6)\cdot\frac{5}{38} + (-1)\cdot\frac{33}{38} = -\frac{3}{38} \approx -7.89\%$$

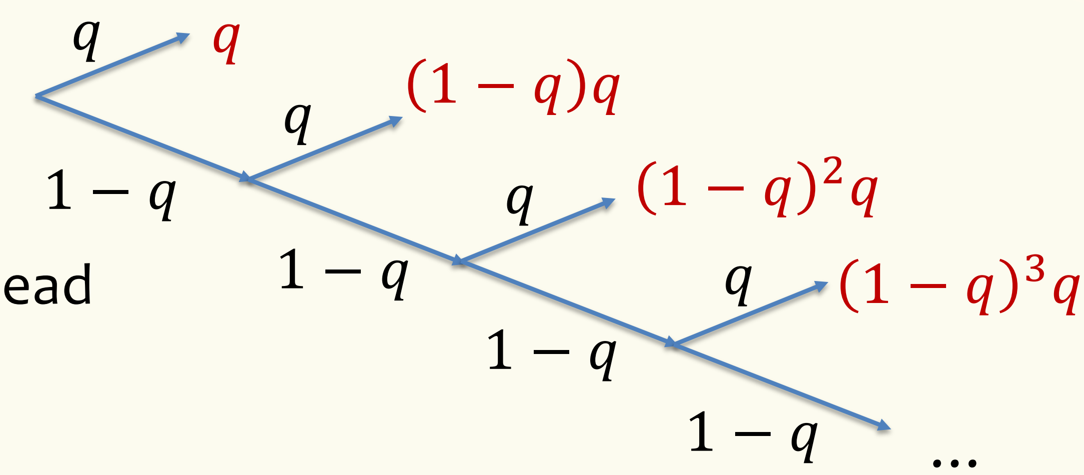
Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- Z = # of coin flips until first head



$$P(Z = i) = q\,(1-q)^{i-1}$$

$$\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot P(Z = i) = \sum_{i=1}^{\infty} i \cdot q(1-q)^{i-1}$$

Converges, so $\mathbb{E}[Z]$ is finite

Can calculate this directly but…

Example – Flipping a biased coin until you see heads

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  $P(H) = q > 0$

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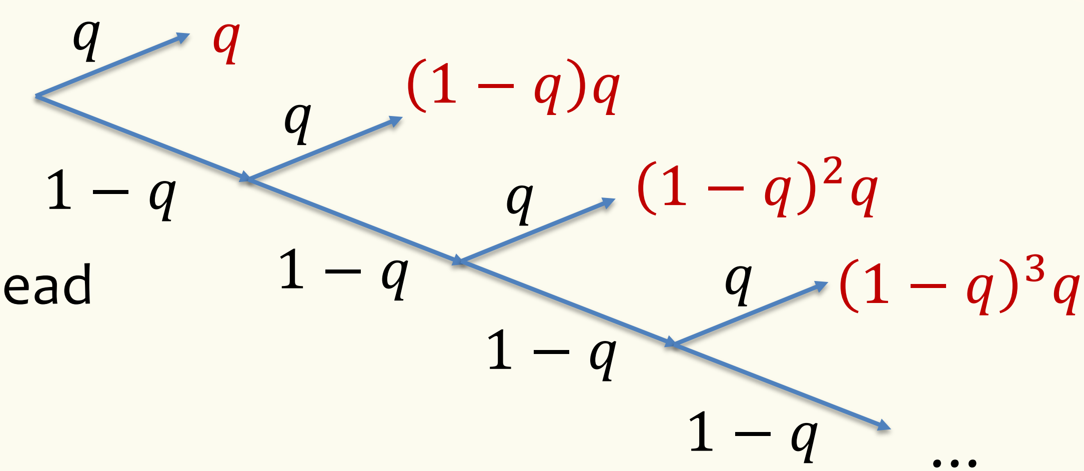

Another view: If you get heads first try you get $Z = 1$;
If you get tails you have used one try and have the same experiment left

$$\mathbb{E}[Z] = q \cdot 1 + (1 - q)(1 + \mathbb{E}(Z))$$

Solving gives $q \cdot \mathbb{E}[Z] = q + (1 - q) = 1$ Implies $\mathbb{E}[Z] = 1/q$

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Next time: Properties of Expectation