CSE 312

Foundations of Computing II

Lecture 5: Conditional Probability and Bayes Theorem

Review Probability

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Review Probability space

Definition. A (discrete) **probability space** is a pair (Ω, P) where:

- Ω is a set called the **sample space.**
- P is the **probability measure,** a function $P: \Omega \to \mathbb{R}$ such that:
  - $P(x) \geq 0$ for all $x \in \Omega$
  - $\sum_{x \in \Omega} P(x) = 1$

Either finite or infinite countable (e.g., integers)

Set of possible **elementary outcomes**

$A \subseteq \Omega$: $P(A) = \sum_{x \in A} P(x)$

prob. 10%

Specify Likelihood (or probability) of each **elementary outcome**

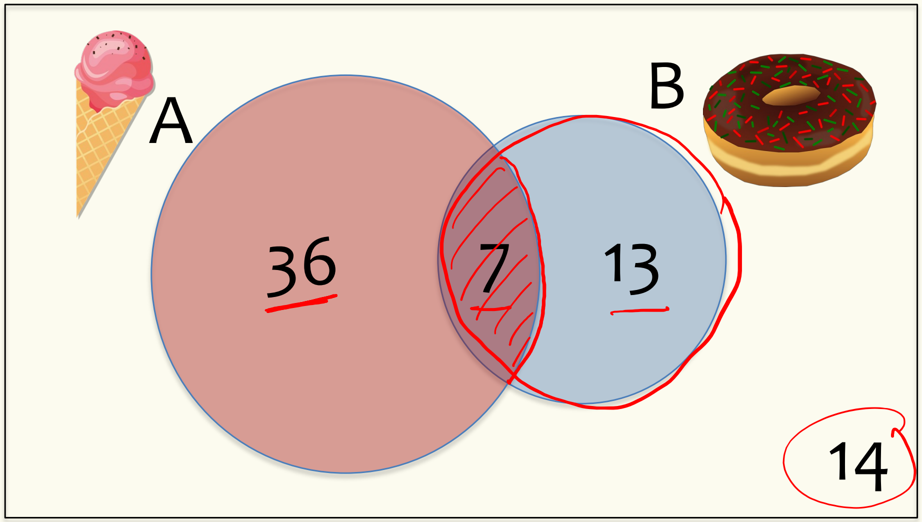
Some outcome must show up. Normalized to sum up to 1.

The likelihood (or probability) of each outcome is non-negative.

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples

Conditional Probability (Idea)



What's the probability that someone likes ice cream **given** they like donuts?

$$\frac{7}{7+13} = \frac{7}{20}$$

Conditional Probability

Definition. The **conditional probability** of event *A* **given** an event *B* happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{\frac{|A \cap B|}{|\Omega|}}{\frac{|B|}{|\Omega|}}$$

An equivalent and useful formula is

$$P(A \cap B) = P(A|B)P(B)$$

$$\frac{|A \cap B|}{|\Omega|} = \frac{|A \cap B|}{|B|} \cdot \frac{|B|}{|\Omega|}$$

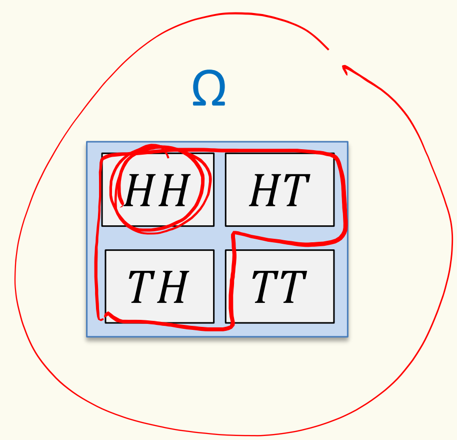
Conditional Probability Examples

Suppose that you flip a fair coin twice.

What is the probability that both flips are heads given that you have at least one head?

Let O be the event that at least *one* flip is heads

Let B be the event that *both* flips are heads



$P(O) = 3/4 \quad P(B) = 1/4 \quad P(B \cap O) = 1/4$

$$P(B|O) = \frac{P(B \cap O)}{P(O)} = \frac{1/4}{3/4} = \frac{1}{3}$$

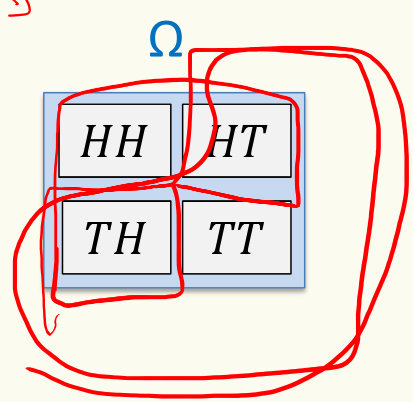
Conditional Probability Examples

Suppose that you flip a fair coin twice.

What is the probability that at least one flip is heads given that at least one flip is tails?

Let H be the event that at least one flip is *heads*

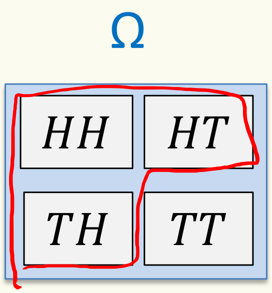
Let T be the event that at least one flip is *tails*



Conditional Probability Examples

Suppose that you flip a fair coin twice.
What is the probability that at least one flip is heads given that at least one flip is tails?

Let H be the event that at least one flip is *heads*
Let T be the event that at least one flip is *tails*



$P(H) = 3/4 \qquad P(T) = 3/4 \qquad P(H \cap T) = 1/2$

$$P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{1/2}{3/4} = \frac{2}{3}$$

Example with Conditional Probability

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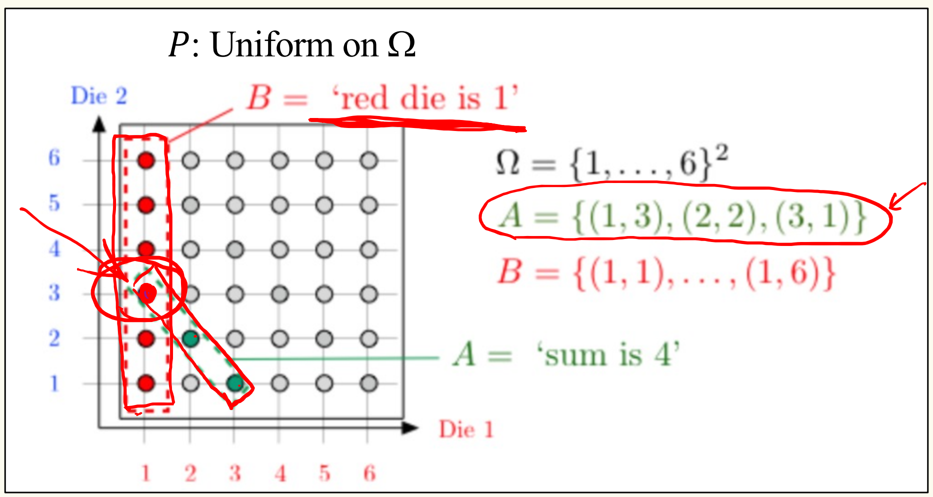
Suppose we toss a red die and a blue die: both 6 sided and all outcomes equally likely.

What is $P(B)$? What is $P(B|A)$?

		$P(B)$	$P(B\|A)$
15	a)	1/6	1/6
45	b)	1/6	1/3
	c)	1/6	3/36
	d)	1/9	1/3



$P(A) = \frac{3}{36} = \frac{1}{12}$

$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{36}}{\frac{1}{12}} = \frac{1}{3}$

$A \cap B = \{(1,3)\}$ $\quad P(A \cap B) = \frac{1}{36}$

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- Bayes Theorem
- Law of Total Probability
- More Examples

Reversing Conditional Probability

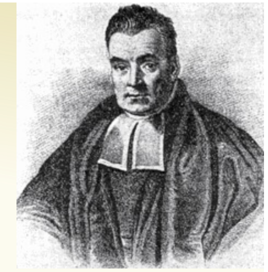
Question: Does $P(A|B) = P(B|A)$?

No!

- Let A be the event you are wet
- Let B be the event you are swimming

$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

Bayes Theorem



A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning B)

Bayes Theorem Proof

Claim:

$$P(A), P(B) > 0 \quad \Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes Theorem Proof

Claim:

$P(A), P(B) > 0 \quad \Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

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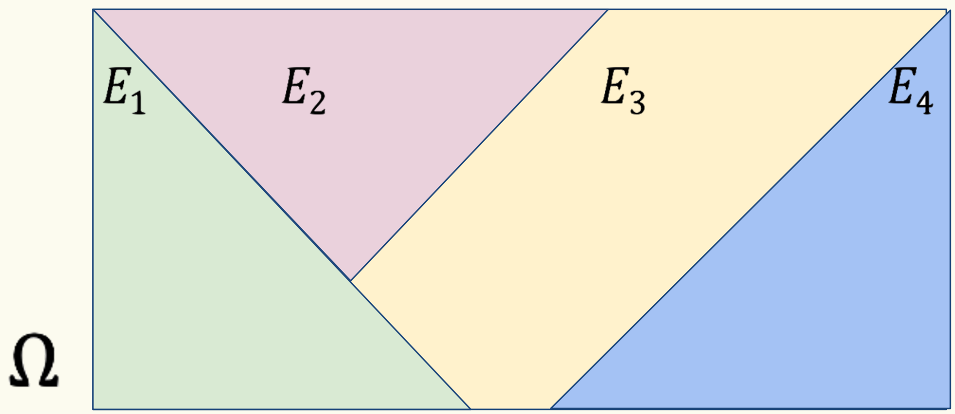
Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



Partition

Definition. Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space Ω if

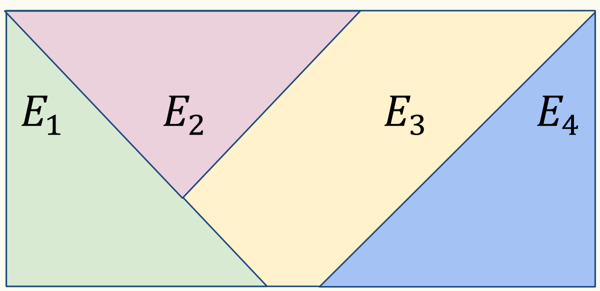
(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

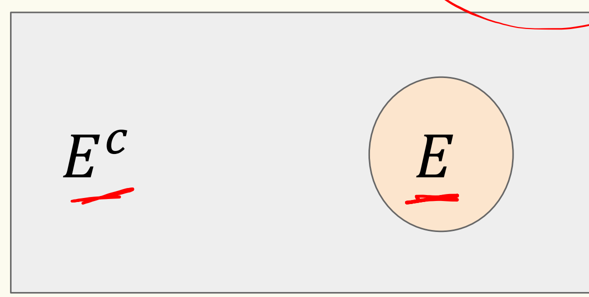
(Pairwise Mutually Exclusive)

$$\forall_i \forall_{i \neq j} \; E_i \cap E_j = \emptyset$$

Ω

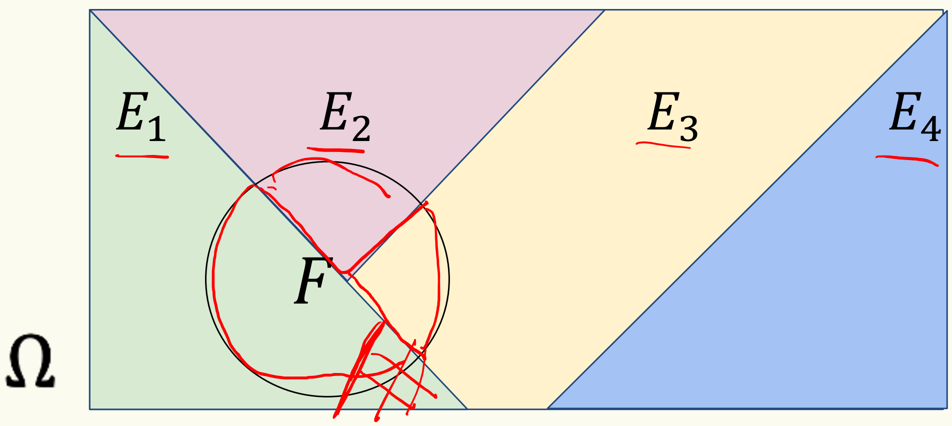


Ω



Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition Ω, what can we say about $P(F)$?



Law of Total Probability (LTP)

Definition. If events $E_1, E_2, \ldots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \ldots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$
We can get the alternate form of this that shows

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

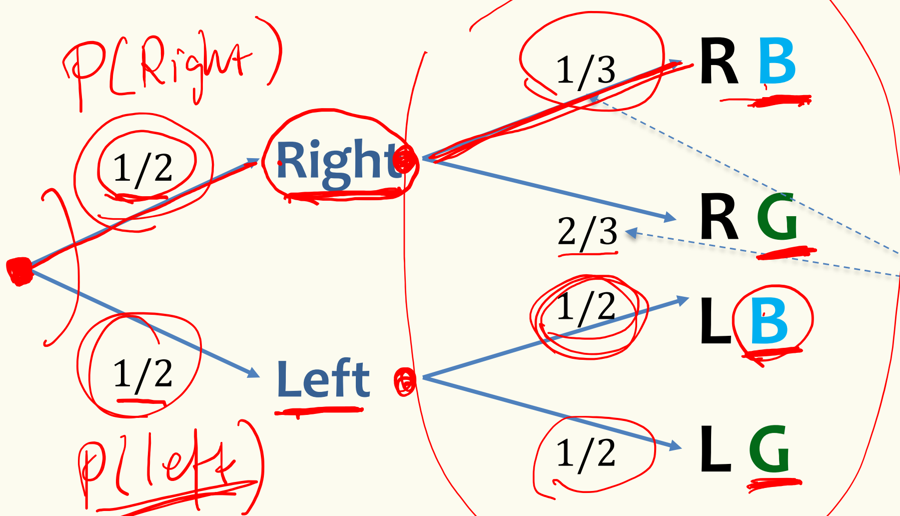
Another Contrived Example

Alice has two pockets:

- **Left pocket:** Two blue balls, two green balls
- **Right pocket:** One blue ball, two green balls.

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]

Sequential Process



- **Left pocket:** Two blue, two green
- **Right pocket:** One blue, two green

1/3 = $P(\text{B}|\text{Right})$ and 2/3 = $P(\text{G}|\text{Right})$

$$P(\text{B}) = P(\text{B} \cap \text{Left}) + P(\text{B} \cap \text{Right})$$

(Law of total probability)

$$= P(\text{Left}) \times P(\text{B}|\text{Left}) + P(\text{Right}) \times P(\text{B}|\text{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

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Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns “positive”, what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

What is the probability you have Zika (event Z) if you test positive (event T).?

$$P(Z|T)$$

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

What is the probability you have Zika (event Z) if you test positive (event T)?

By Bayes Rule, $P(Z|T) = \dfrac{P(T|Z)P(Z)}{P(T)}$

By the Law of Total Probability, $P(T) = P(T|Z)P(Z) + P(T|Z^c)P(Z^c)$

$$= \frac{98}{100} \cdot \frac{5}{1000} + \frac{1}{100} \cdot \frac{995}{1000} = \frac{490}{100000} + \frac{995}{100000}$$

What is the probability that you do not have Zika (event Z^c)?

$P(Z^c) = 1 - P(Z) = 99.5\%$

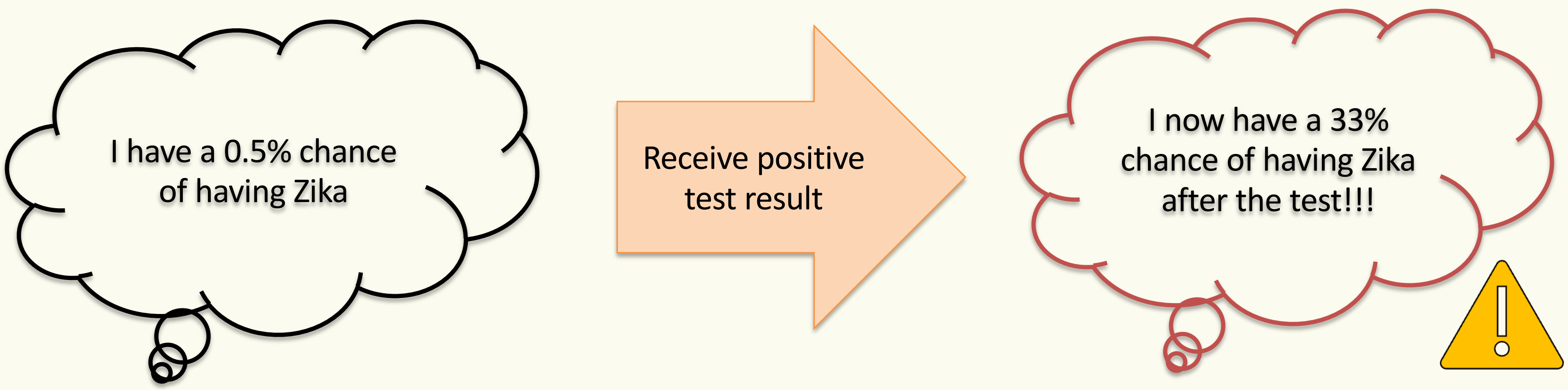
So, $P(Z|T) \approx 33\ \%$

Philosophy – Updating Beliefs

Your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: $P(Z)$

Posterior: $P(Z|T)$

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

What is the probability you have Zika (event Z) if you test negative (event T^c)?

By Bayes Rule, $P(Z|T^c) = \dfrac{P(T^c|Z)P(Z)}{P(T^c)}$

By the Law of Total Probability, $P(T^c) = P(T^c|Z)P(Z) + P(T^c|Z^c)P(Z^c)$

$$= \frac{2}{100} \cdot \frac{5}{1000} + \left(1 - \frac{1}{100}\right) \cdot \frac{995}{1000} = \frac{10}{100000} + \frac{98505}{100000}$$

What is the probability you test negative (event T^c) if you have Zika (event Z)?

$P(T^c|Z) = 1 - P(T|Z) = 2\%$

So, $P(Z|T^c) = \frac{10}{10+98505} \approx 0.01\ \%$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

We just used this implicity on the negative Zika test example with $E = Z$ and $F = T^c$

Simple Partition: In particul... probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Conditional Probability Defines a Probability Space

The probability conditioned on $\mathcal{A}$ follows the same properties as (unconditional) probability.

Example. $P(\mathcal{B}^c|\mathcal{A}) = 1 - P(\mathcal{B}|\mathcal{A})$

Formally. (Ω, P) is a probability space and $P(\mathcal{A}) > 0$

➡ $(\mathcal{A}, P(\cdot\,|\mathcal{A}))$ is a probability space