CSE 312

Foundations of Computing II

Lecture 2: Combinations and Binomial Coefficients

Announcements

Homework:

- Pset1 was posted on Wednesday and is due 11:59pm next Wednesday.

- We will have the same pattern for the other assignments, excluding the week of the midterm and the last two assignments after Thanksgiving, which will be due on Fridays.

Python programming on homework:

- Some problem sets will include coding problems
  - in Python (*no prior knowledge or experience required*)
  - provide a deeper understanding of how theory we discuss is used in practice
  - should be fun

Announcements

- **Problem Set 1**
  - Read the first page for how to write up your homework solutions. Don't wait until you are working on the questions to figure it out!
  - Section solutions are a good place to look at for examples.

- **Resources**
  - Textbook readings can provide another perspective
  - Theorems & Definitions sheet – https://www.alextsun.com/files/defs_thms.pdf
  - Office Hours
  - EdStem discussion

- **EdStem discussion etiquette**
  - OK to publicly discuss content of the course and any confusion over topics discussed in class, but **not *solutions*** for current homework problems, or anything about current exams that have not yet been graded.
  - It is also acceptable to ask for clarifications about what current homework problems are asking and concepts behind them, just not about their solutions.

Quick counting summary from last class

- **Sum rule:**

  If you can choose from

  - EITHER one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- **Product rule:**

  In a sequential process, if there are

  - n_1 choices for the 1^{st} step,
  - n_2 choices for the 2^{nd} step (given the first choice), ..., and
  - n_k choices for the k^{th} step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times n_3 \times \cdots \times n_k$

- Representation of the problem is important (creative part)

k-permutations

Fact. # of k-element sequences of distinct symbols from an n-element set is

$$P(n,k) = n\times(n-1)\times\cdots\times(n-k+1) = \frac{n!}{(n-k)!}$$

"How many sequences of 5 ***distinct*** *alphabet letters from* $\{A, B, \dots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26\times25\times24\times23\times22 = 7893600$

k-combination

Fact. The number of subsets of size k of a set of size n is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

a.k.a. Binomial coefficient (verbalized as "n choose k")

Notation: $\binom{S}{k}$ = set of all k-element subsets of S. $\left|\binom{S}{k}\right| = \binom{|S|}{k}$

[also called **combinations**]

Quick Summary

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule ➔ n^k
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof,
i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

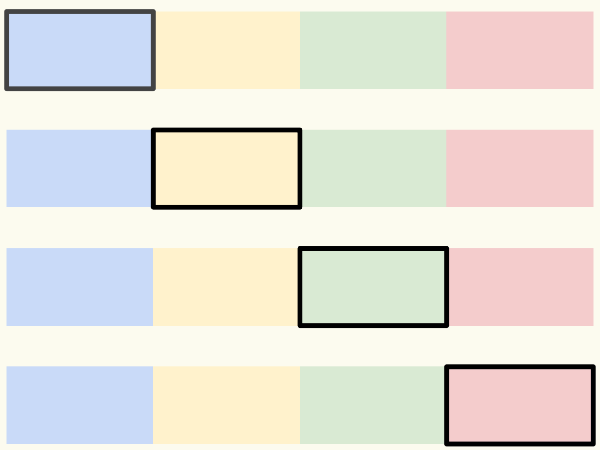
Why??

Symmetry in Binomial Coefficients – A different proof

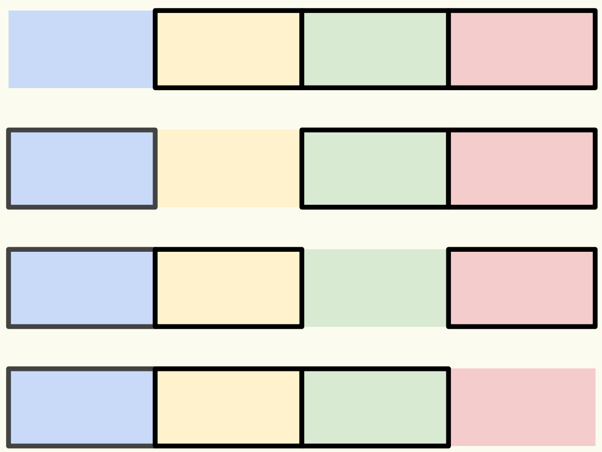
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

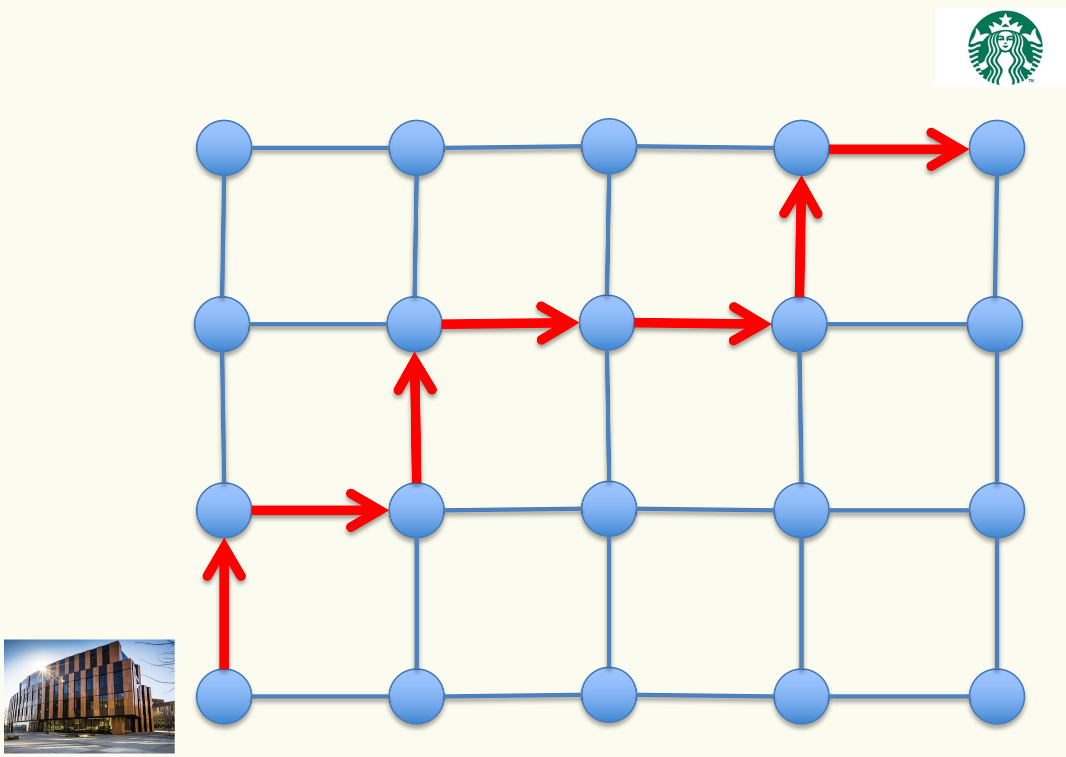
1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$

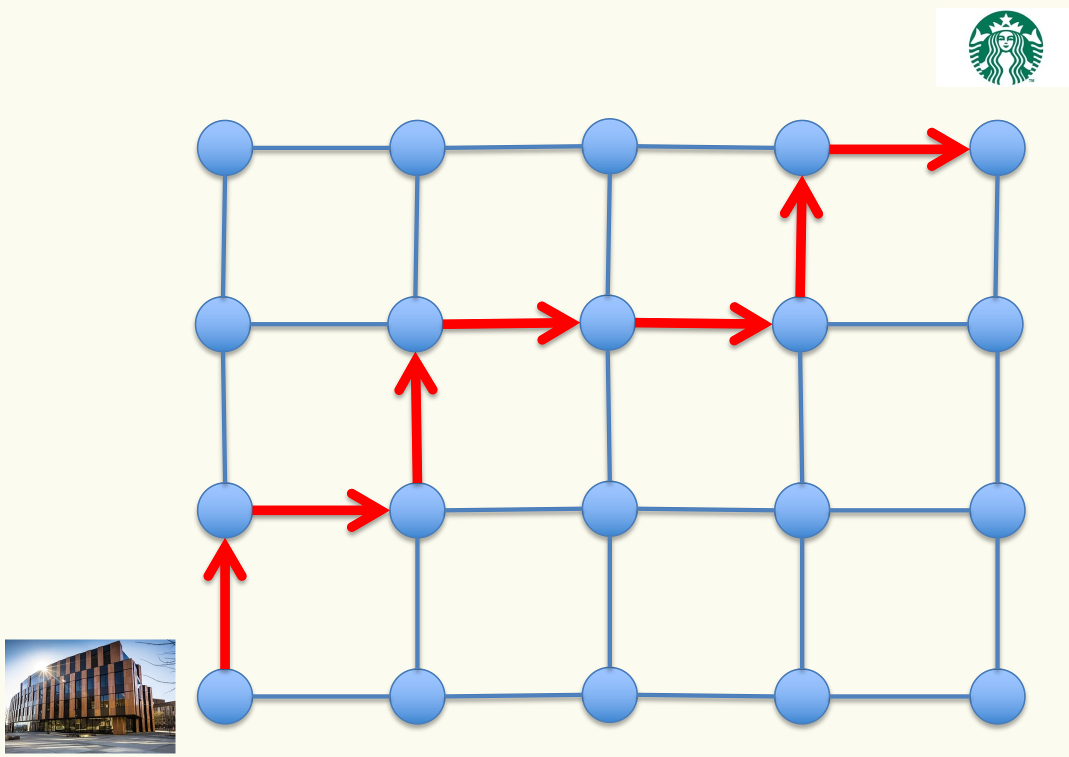


Example – Counting Paths



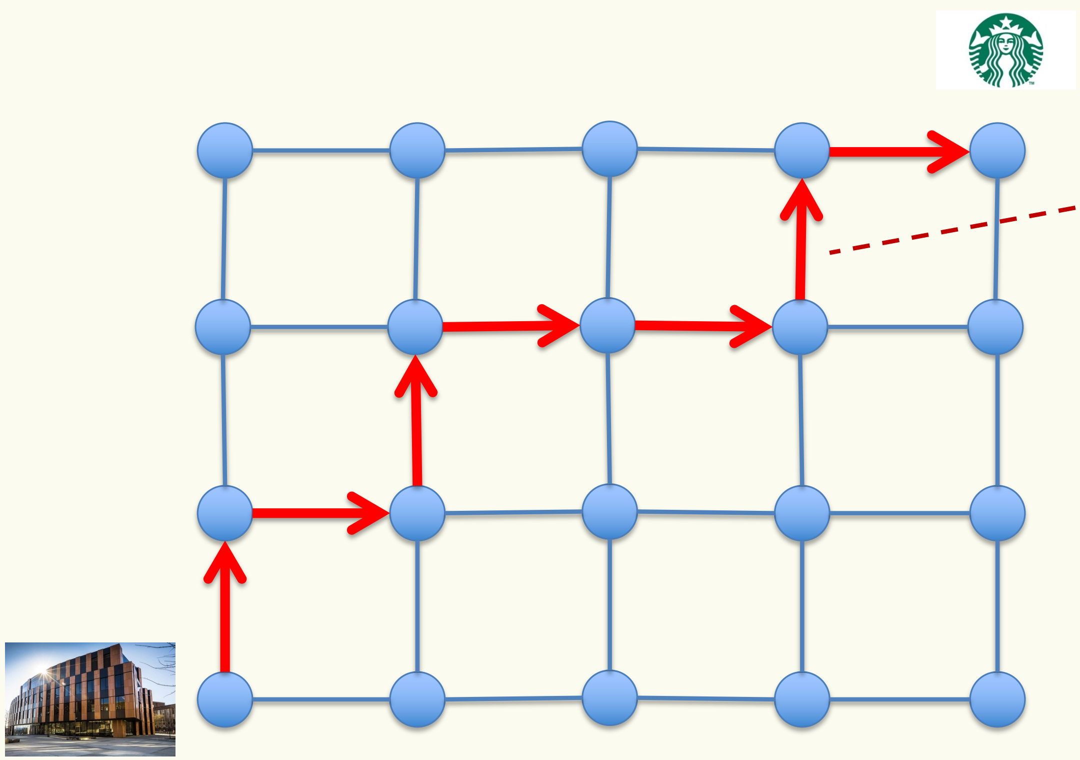
"How many shortest paths from Gates to Starbucks?"

Example – Counting Paths



How do we represent a shortest path?

Example – Counting Paths



Path $\in \{\uparrow, \rightarrow\}^7$

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

\# ↑'s =3, # →'s=4

Poll: # of shortest paths?

A. 2^7

B. $\frac{7!}{4!}$

C. $\binom{7}{4} = \frac{7!}{4!3!}$

D. *No idea*

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Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

Hint: we can represent each solution as a binary string.

Example – Sum of integers

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Clever representation of solutions

$(3,1,1)$	$(2,1,2)$	$(1,0,4)$
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1

Example – Sum of integers

Example: $k = 3$, $n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1) \rightarrow 1\,1\,1\,0\,1\,0\,1$

$(2,1,2) \rightarrow 1\,1\,0\,1\,0\,1\,1$

$(1,0,4) \rightarrow 1\,0\,0\,1\,1\,1\,1$

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

After a change in representation, the problem magically reduces to counting combinations.

Example – Sum of integers

Example: $k = 3, n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

(3,1,1)	(2,1,2)	(1,0,4)
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1
* * * | * | *	* * | * | * *	* | | * * * *

Also called the "stars and bars" method

*aka. "stars and bars method"

More general counting using binary encoding*

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Example with k non-negative integers summing to n: bins are the k integers, balls are the n 1's that add to n.

Coins

How many ways can you distribute 32 identical coins among Alex, Barbara, Charlie, Dana, and Eve?

1. Identify balls
2. Identify bins

$$\binom{32+5-1}{5-1}$$

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Guess: 7!   Correct?!

No! e.g., swapping two T's also leads to *SEATTLE*
swapping two E's also leads to *SEATTLE*

Counted as separate permutations, but they lead to the same word.

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1$$

Locations of T's

Locations of E's

Location of S

Location of A

Location of L

T	E	S	T	A	L	E

Another way to look at SEATTLE

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2!\,\cancel{5!}} \times \frac{\cancel{5!}}{2!\,\cancel{3!}} \times \cancel{3!}$$

$$= \frac{7!}{2!\,2!} = 1260$$

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T's, and divide by 2! again for 2 duplicate E's.

More generally…

How many ways can you arrange the letters in "Godoggy"?

$n = 7$ Letters, $k = 4$ Types {G, O, D, Y}

$n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 1$

$$\frac{7!}{3!2!1!1!} = \binom{7}{3,2,1,1}$$

Multinomial coefficients

Multinomial Coefficients

If we have k types of objects ($\boldsymbol{n}$ total), with $\boldsymbol{n_1}$ of the first type, $\boldsymbol{n_2}$ of the second, …, and $\boldsymbol{n_k}$ of the k^{th}, then the number of orderings possible is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Binomial Coefficients – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ — Symmetry in Binomial Coefficients

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ — Follows from Binomial Theorem

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ — Pascal's Identity (Next lecture)

Binomial Theorem: Idea

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

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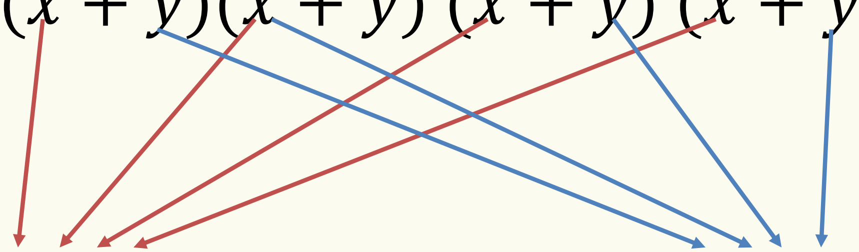
$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x + y)^n = (x + y) \dots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x + y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x + y)$ (the other $n - k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Apply with $x = y = 1$

Many properties of sums of binomial coefficients can be found by plugging in different values of x and y in the Binomial Theorem.

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Recall: Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

Format for a combinatorial argument/proof of $a = b$

- Let S be a set of objects
- Show how to count $|S|$ one way $\Rightarrow$ $|S| = a$
- Show how to count $|S|$ another way $\Rightarrow$ $|S| = b$

Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ ...$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

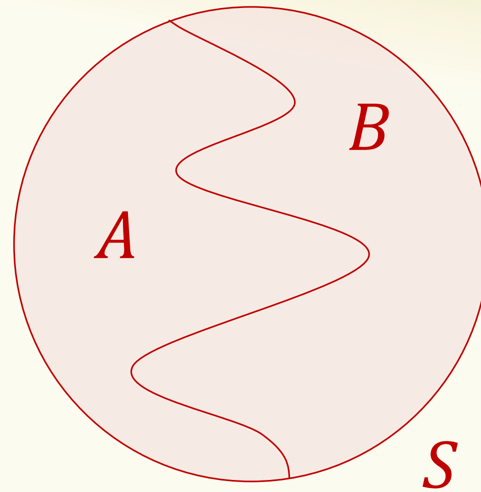
Hard work and not intuitive

Let's see a combinatorial argument

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



$S = A \cup B$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \;=\; |A| \;+\; |B|$

Combinatorial proof idea:
- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

$|S| = \binom{n}{k}$

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

e.g. $n = 4, k = 2$, $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n]$ that DO include n

$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$

B: set of size k subsets of $[n]$ that DON'T include n

$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

A: set of size k subsets of $[n]$ that DO include n

n is in set, need to choose other $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

B: set of size k subsets of $[n]$ that DON'T include n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$