CSE 312

Foundations of Computing II

Lecture 2: Combinations and Binomial Coefficients

Announcements

Homework:

- Pset1 was posted on Wednesday and is due 11:59pm next Wednesday.
- Read the first page for how to write up your homework solutions. Don't wait until you are working on the questions to figure it out!
- We will have python programming tasks later (4~5 of them)

Resources

- Textbook readings can provide another perspective
- Theorems & Definitions sheet – https://www.alextsun.com/files/defs_thms.pdf
- Office Hours
- EdStem discussion

Quick counting summary from last class

- **Sum rule:**

  If you can choose from

  - EITHER one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- **Product rule:**

  In a sequential process, if there are

  - n_1 choices for the 1^{st} step,
  - n_2 choices for the 2^{nd} step (given the first choice), ..., and
  - n_k choices for the k^{th} step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times n_3 \times \cdots \times n_k$

- Representation of the problem is important (creative part)

Quick Summary

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule → n^k
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation → $\frac{n!}{(n-k)!}$

n × n-1 × n-2 . . × (n-k+1)

- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination → $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Number of Subsets

“How many size-5 ***subsets*** *of* $\{A, B, \ldots, Z\}$*?”*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Difference from k-permutations: NO ORDER

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

Number of Subsets – Idea

Consider a sequential process:

1. Choose a subset $S \subseteq \{A, B, \ldots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

$$??? \times 5! = \frac{26!}{21!}$$

Outcome: A sequence of 5 distinct letters from $\{A, B, \ldots, Z\}$

$$??? = \frac{26!}{21!\,5!} = 65780$$

k-combination

Fact. The number of subsets of size k of a set of size n is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

a.k.a. Binomial coefficient (verbalized as "n choose k")

Notation: $\binom{S}{k}$ = set of all k-element subsets of S.

[also called **combinations**]

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof, i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

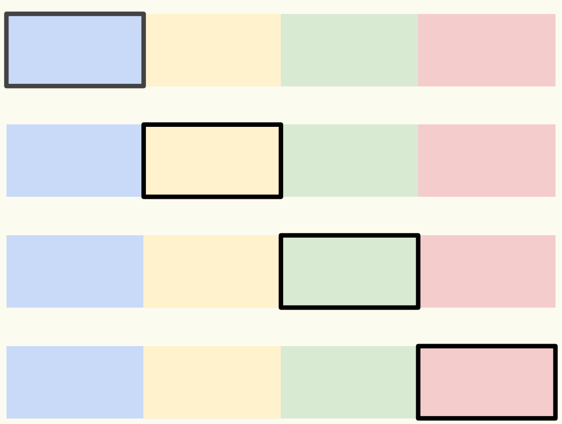
Why??

Symmetry in Binomial Coefficients – A different proof

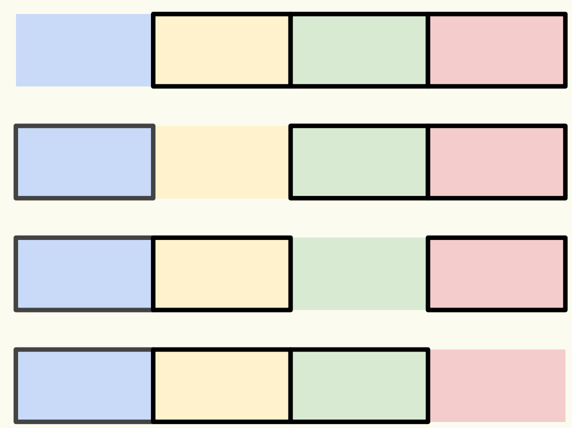
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

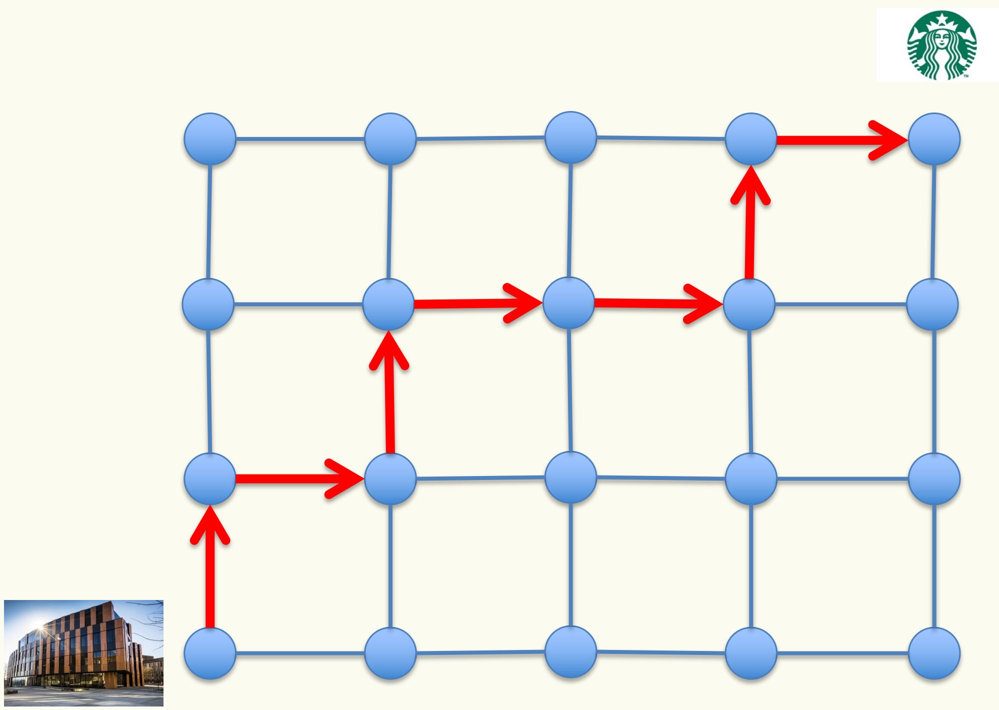
1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$

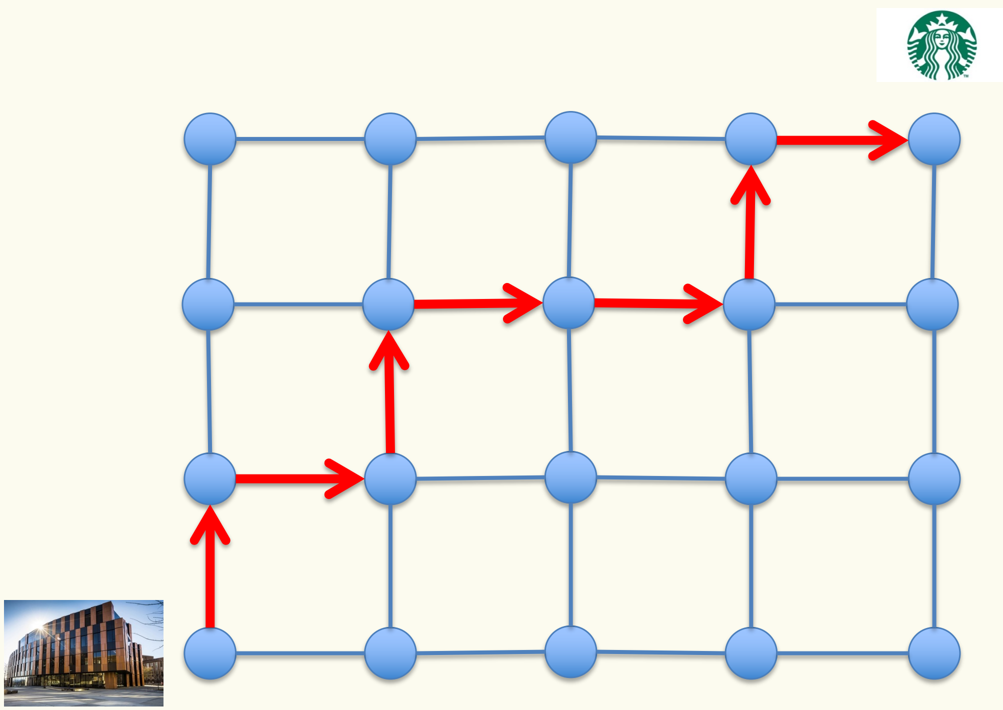


Example – Counting Paths



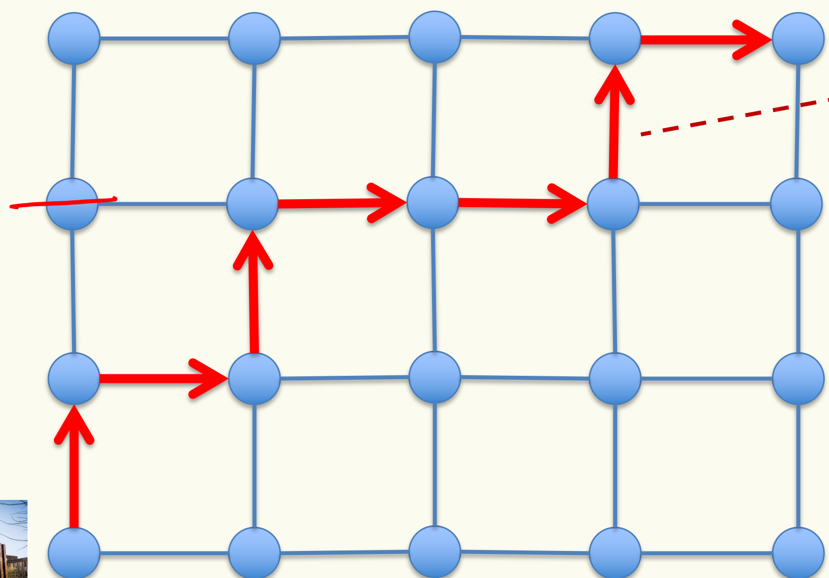
"How many shortest paths from Gates to Starbucks?"

Example – Counting Paths



How do we represent a shortest path?

Example – Counting Paths



Path $\in \{\uparrow, \rightarrow\}^7$

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

$\#\uparrow$'s $=3$, $\#\rightarrow$'s$=4$

Poll: # of shortest paths?

A. 2^7

B. $\frac{7!}{4!}$

C. $\binom{7}{4} = \frac{7!}{4!3!}$

D. No idea

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Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

Poll: # of solutions?

A. 6^3

B. $\binom{7}{2} = \frac{7!}{2!5!}$

C. $\binom{7}{3} = \frac{7!}{4!3!}$

D. *No idea*

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Hint: we can represent each solution as a binary string.

Example – Sum of integers

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Clever representation of solutions

$(3,1,1)$ → $1\ 1\ 1\ 0\ 1\ 0\ 1$

$(2,1,2)$ → $1\ 1\ 0\ 1\ 0\ 1\ 1$

$(1,0,4)$ → $1\ 0\ 0\ 1\ 1\ 1\ 1$

Example – Sum of integers

Example: $k = 3$, $n = 5$

\# sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$ → $1\ 1\ 1\ 0\ 1\ 0\ 1$

$(2,1,2)$ → $1\ 1\ 0\ 1\ 0\ 1\ 1$

$(1,0,4)$ → $1\ 0\ 0\ 1\ 1\ 1\ 1$

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

\# sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

After a change in representation, the problem magically reduces to counting combinations.

Example – Sum of integers

Example: $k = 3, n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$	$(2,1,2)$	$(1,0,4)$
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1
* * * | * | *	* * | * | * *	* | | * * * *

Also called the "stars and bars" method

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Guess: 7! Correct?!

No! e.g., swapping two T's also leads to *SEATTLE*
swapping two E's also leads to *SEATTLE*

Counted as separate permutations, but they lead to the same word.

A mixed example – Word Permutations (aka Anagrams)

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Location of S

Location of A

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1$$

T	E	S	T	A	L	E

Location of L

Locations of E's

Locations of T's

Another way to look at SEATTLE

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2!\,5!} \times \frac{5!}{2!\,3!} \times 3!$$

$$= \frac{7!}{2!\,2!} = 1260$$

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T's, and divide by 2! again for 2 duplicate E's.

More generally…

How many ways can you arrange the letters in "Godoggy"?

$n = 7$ Letters, $k = 4$ Types $\{G, O, D, Y\}$

$n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 1$

$$\frac{7!}{3!2!1!1!} = \binom{7}{3,2,1,1}$$

Multinomial coefficients

Multinomial Coefficients

If we have k types of objects ($\boldsymbol{n}$ total), with $\boldsymbol{n_1}$ of the first type, $\boldsymbol{n_2}$ of the second, …, and $\boldsymbol{n_k}$ of the k^{th}, then the number of orderings possible is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Binomial Coefficients – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ **Symmetry in Binomial Coefficients**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ **Follows from Binomial Theorem**

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

Binomial Theorem: Idea

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$

$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Poll: What is the coefficient for xy^3?

A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

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Binomial Theorem: Idea

$$(x+y)^n = (x+y)\ldots(x+y)$$

Each term is of the form $x^k y^{n-k}$, since each term is made by multiplying exactly n variables, either x or y, one from each copy of $(x+y)$

How many times do we get $x^k y^{n-k}$?

The number of ways to choose x from exactly k of the n copies of $(x+y)$ (the other $n-k$ choices will be y) which is:

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Apply with $x = y = 1$

Many properties of sums of binomial coefficients can be found by plugging in different values of x and y in the Binomial Theorem.

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

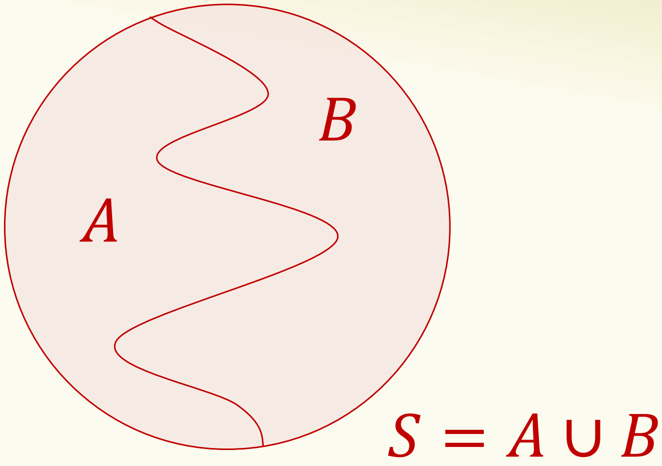
Hard work and not intuitive

Let's see a combinatorial argument

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |B| \quad + \quad |A|$$



Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |B| \quad + \quad |A|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

$|S| = \binom{n}{k}$

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

e.g. $n = 4, k = 2$, $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n-1]$ (i.e., DON'T include n)

$A = \{\{1,2\}, \{1,3\}, \{2,3\}\}$

B: set of size $k-1$ subsets of $[n-1]$ that (i.e., DO include n)

$B = \{\{1,4\}, \{2,4\}, \{3,4\}\}$

Example – Pascal's Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |B| \quad + \quad |A|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

A: set of size k subsets of $[n-1]$ (i.e., DON'T inc

B: set of size $k-1$ subsets of $[n-1]$ that (i.e., I

n not in set, need to choose k elements from $[n-1]$

$|B| = \binom{n-1}{k}$

n is in set, need to choose other $k-1$ elements from $[n-1]$

$|A| = \binom{n-1}{k-1}$